Exercise-1

✍ **Marked questions are recommended for Revision.**
✍ चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : SUBJECTIVE QUESTIONS

भाग - I : विषयात्मक प्रश्न (SUBJECTIVE QUESTIONS)

Section (A) : Definite Integration in terms of Indefinite Integration, using substitution and By parts

खण्ड (A) : प्रतिस्थापन तथा खण्डशः समाकलन की सहायता से अनिश्चित समाकलन के रूप में निश्चित समाकलन

A-1. Evaluate :
मान ज्ञात कीजिए–

(i) $\int_0^1 \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}\,dx$ (ii)✍ $\int_0^1 x\cos(\tan^{-1}x)\,dx$

Ans. (i) $-\frac{10}{21}$ (ii) $\sqrt{2}-1$

Sol. (i) $I = \int_0^1 \left[x^{\frac{2}{3}-\frac{1}{2}} - x^{\frac{1}{4}-\frac{1}{2}}\right] dx = \int_0^1 \left(x^{1/6} - x^{-1/4}\right) dx$

(ii) $\cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$, Hence integral is अतः समाकल $\int_0^1 \frac{x}{\sqrt{1+x^2}}dx = \sqrt{1+x^2}\Big|_0^1 = \sqrt{2}-1$

A-2. Evaluate :
मान ज्ञात कीजिए–

(i) $\int_{-\infty}^{\infty} \frac{dx}{x^2+2x+2}$ (ii) $\int_{\sqrt{2}}^{\infty} \frac{dx}{x\sqrt{x^2-1}}$ (iii) $\int_0^4 \frac{x^2}{1+x}\,dx$

(iv) $\int_0^{\pi/2} \sqrt{\cos\theta}\,\sin^3\theta \quad d\theta$

Ans. (i) π (ii) $\frac{\pi}{4}$ (iii) 4 + ℓn 5 (iv) $\frac{8}{21}$

Sol. (i) $I = \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2+1} = \left[\tan^{-1}(x+1)\right]_{-\infty}^{\infty} = \frac{\pi}{2}-\left(-\frac{\pi}{2}\right) = \pi$

(ii) $x = \sec\theta \Rightarrow \quad dx = \sec\theta\tan\theta\,d\theta \Rightarrow I = \int_{\pi/4}^{\pi/2} \frac{\sec\theta \quad \tan\theta \quad d\theta}{\sec\theta \;.\; \tan\theta}\,dx = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$

(iii) $I = \int_0^4 \frac{x^2+1-1}{1+x}\,dx = \int_0^4 (x-1)\,dx + \int_0^4 \frac{1}{1+x}\,dx = \left[\frac{x^2}{2}-x\right]_0^4 + \left[\ell n\ (1+x)\right]_0^4$

$= 4 + \ell n5$

(iv) Let माना $\cos\theta = t$

$-\sin\theta\,d\theta = dt$

$I = -\int_1^0 \sqrt{t}(1-t^2)\ .\ dt$; $I = -\left[\frac{t^{3/2}}{3/2}-\frac{t^{7/2}}{7/2}\right]_1^0$

$I = -\left(-\frac{2}{3}+\frac{2}{7}\right) = -\left(\frac{-14+6}{21}\right)$; $I = +\frac{8}{21}$

A-3. Evaluate :

मान ज्ञात कीजिए–

(i) $\int_0^1 \sin^{-1}x \ \ dx$ (ii) $\int_1^2 \frac{\ell n \ x}{x^2} dx$ (iii) $\int_0^1 x^2 \sin^{-1}x \ \ dx$.

Ans. (i) $\frac{\pi-2}{2}$ (ii) $\frac{1}{2}\ell n\left(\frac{e}{2}\right)$ (iii) $\frac{\pi}{6}-\frac{2}{9}$

Sol. (i) $I = \int_0^1 1.\sin^{-1}x \ \ dx$ $I = \left[x\sin^{-1}x - \int\frac{x \ \ dx}{\sqrt{1-x^2}}\right]_0^1 = \frac{\pi-2}{2}$

(ii) $I = \int_1^2 \frac{\log x}{x^2} \ dx$

By parts log x $\to$ I खण्डश: समाकलन से log x $\to$ I

$\frac{1}{x^2} \to$ II

On solving हल करने पर $\Rightarrow$ $I = \frac{1}{2}\ell n\frac{e}{2}$

(iii) $I = \int_0^1 \underset{\text{II}}{\frac{x^2}{}} \ \underset{\text{I}}{\frac{\sin^{-1}x}{}} \ dx$ $\Rightarrow$ $I = \left[\sin^{-1}x\frac{x^3}{3}\right]_0^1 - \frac{1}{3}\int_0^1\frac{x^3}{\sqrt{1-x^2}} \ dx = I_1 - \frac{1}{3}I_2$

I_2 Let मानाकि $1 - x^2 = t^2$ $\Rightarrow$ $-2x \ dx = 2t \ dx$

$\Rightarrow$ $x \ dx = -t \ dt$ $\Rightarrow$ $I_2 = \int_0^1(1-t^2) \ dt = \left[t-\frac{t^3}{3}\right]_0^1 = \frac{2}{3}$ & $I_1 = \frac{\pi}{6}$ $\Rightarrow$ $I = \frac{\pi}{6} - \frac{2}{9}$

A-4. Evaluate मान ज्ञात कीजिए

(i) $\int_0^{\pi/3} f(x) \ dx$ where f(x) = Minimum {tanx, cot x} $\forall$ x $\in$ $\left(0, \frac{\pi}{2}\right)$

(ii) $\int_{-1}^{1}$ f(x) dx where f(x) = min $\{x+1, \sqrt{1-x}\}$

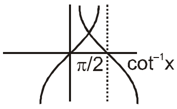
(iii)✍ $\int_{-1}^{1} f(x) \ \ dx$ where f(x) = minimum (|x| , 1 – |x|, 1/4)

Hindi (i) $\int_0^{\pi/3} f(x) \ dx$ जहाँ f(x) = न्यूनतम {tanx, cot x} $\forall$ x $\in$ $\left(0, \frac{\pi}{2}\right)$

(ii) $\int_{-1}^{1}$ f(x) dx जहाँ f(x) = न्यूनतम $\{x+1, \sqrt{1-x}\}$

(iii)✍ $\int_{-1}^{1} f(x) \ \ dx$ जहाँ f(x) = न्यूनतम (|x| , 1 – |x|, 1/4)

Ans. **(i)** $\ell n \ (\sqrt{3})$ (ii) 7/6 (iii) $\frac{3}{8}$



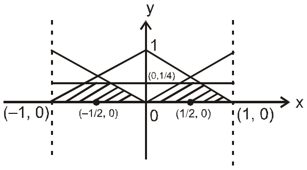
Sol. (i) $I = \int_0^{\pi/3} f(x) \ \ dx = \int_0^{\pi/4} \tan x \ \ dx + \int_{\pi/4}^{\pi/3} \cot x \ \ dx = \ell n\left(\sqrt{3}\right)$

(ii) f(x) = min (निम्नतम) $\left\{ x+1, \sqrt{1-x} \ \right\} = \begin{cases} x+1 & -1<x<0 \\ \sqrt{1-x} & 0<x<1 \end{cases}$

$$\therefore \quad \int_{-1}^{1} f(x)\ dx = \left[\int_{-1}^{0} (x+1)\ dx + \int_{0}^{1} \sqrt{1-x}\ dx\right]$$

$$= \left[\left(\frac{x^2}{2}+x\right)\Bigg|_{-1}^{0} - \frac{2}{3}(1-x)^{3/2}\Bigg|_{0}^{1}\right] = \left[0-\left(\frac{1}{2}-1\right)-\frac{2}{3}(0-1)\right] = \left(\frac{1}{2}+\frac{2}{3}\right) = \frac{7}{6}$$

(iii) $$\int_{-1}^{1} f(x)\ dx = 2\int_{0}^{1} f(x)\ dx = 2\left(\int_{0}^{1/4} x\ dx + \int_{1/4}^{3/4} \frac{1}{4}dx + \int_{3/4}^{1} (1-x)\ dx\right)$$



$$= 2\left(\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{4}\times\frac{1}{4}\right) = \frac{3}{8}$$

A-5. Evaluate

मान ज्ञात कीजिए–

(i) $\displaystyle\int_{0}^{1} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$ (ii) $\displaystyle\int_{0}^{1} \frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\ dx$

(iii) $\displaystyle\int_{a}^{b} \sqrt{(x-a)(b-x)}\ dx$, a > b (iv) $\displaystyle\int_{0}^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right) dx$

Ans. (i) $\frac{\pi}{2} - \ell n\, 2$ (ii) $\frac{4-\pi}{4\sqrt{2}}$ (iii) $-\frac{\pi}{8}\ (b-a)^2$ (iv) $\pi\left(1-\frac{1}{\sqrt{3}}\right) - \ell n\, 4$

Sol. (i) $$I = \int_{0}^{1} \sin^{-1}\left(\frac{2x}{1+x^2}\right) dx$$

Let (माना) $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\Rightarrow I = \int_{0}^{\pi/4} 2\theta\ \sec^2\theta\ d\theta = 2\left[\theta\ \tan\theta\right]_0^{\pi/4} - 2\int_{0}^{\pi/4} \tan\theta\ d\theta = 2\left(\frac{\pi}{4}\right) - 2\left[\ell n\ \sec\theta\right]_0^{\pi/4}$$

$$= \frac{\pi}{2} - 2\ell n\sqrt{2} = \frac{\pi}{2} - \ell n 2$$

(ii) Let (माना) $\tan^{-1} x = t \Rightarrow \frac{dx}{1+x^2} = dt$

$$\therefore \quad I = \int_{0}^{\pi/4} \frac{t\ \tan\ t\ \cdot\ dt}{\sqrt{1+\tan^2 t}} = \int_{0}^{\pi/4} t\ \cdot\ \sin t\ \ dt = -\frac{\pi}{4\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4-\pi}{4\sqrt{2}}$$

(iii) $$I = \int_{a}^{b} \sqrt{(x-a)(b-x)}\ dx$$

put $x = a\sin^2\theta + b\cos^2\theta$ रखने पर

$$I = 2\int_{0}^{\pi/2} -(b-a)^2\ \sin^2\theta\cos^2\theta\ d\theta\ ;\ I = -2\,(b-a)^2 \int_{0}^{\pi/2} \left(\frac{\sin 2\theta}{2}\right)^2 d\theta$$

$$I = -\frac{(b-a)^2}{2}\int_{0}^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right) d\theta\ ; \quad I = -\frac{(b-a)^2}{4}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = -\frac{(b-a)^2\pi}{8}$$

(iv) $I = \int_0^{\sqrt{3}} \tan^{-1}\left(\frac{2x}{1-x^2}\right)\ dx$

put x = tan θ ∈ $\left(0, \frac{\pi}{3}\right)$, 2θ ∈ $\left(0, \frac{2\pi}{3}\right)$ रखने पर

$$I = \int_0^{\pi/3} \tan^{-1}(\tan 2\theta)\ \sec^2\theta d\theta = \int_0^{\pi/4} 2\theta\ \sec^2\theta\ d\theta + \int_{\pi/4}^{\pi/3} (2\theta-\pi)\ \sec^2\theta\ d\theta$$

$$= \int_0^{\pi/3} 2\theta\sec^2\theta\ d\theta - \pi \int_{\pi/4}^{\pi/3} \sec^2\theta\ d\theta = \left(\frac{2\pi}{\sqrt{3}} - \ell n4\right) - \pi\ (\tan\theta)_{\pi/4}^{\pi/3}$$

$I = \pi\left(1-\frac{1}{\sqrt{3}}\right) - \ell n\ 4.$ **Ans.**

A-6. Evaluate :
मान ज्ञात कीजिए–

(i) $\int_0^{\infty} \frac{dx}{e^x + e^{-x}}$ (ii) $\int_0^1 \frac{x}{1+\sqrt{x}}\ dx$ (iii) $\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\ dx$

(iv) $\int_0^{\pi/2} \frac{\sin 2\theta\ \ d\theta}{\sin^4\theta + \cos^4\theta}$ (v) $\int_0^{\pi/4} \frac{\sin x + \cos x}{9 + 16\ \ \sin 2x} dx$

Ans. (i) $\frac{\pi}{4}$ (ii) $\frac{5}{3} - 2\ \ell n2$ (iii) $\ell n\left(\frac{9}{8}\right)$ (iv) $\frac{\pi}{2}$ (v) $\frac{1}{20}\ \ \ell n\ \ 3$

Sol. (i) $\int_0^{\infty} \frac{dx}{e^x + e^{-x}} = \int_0^{\infty} \frac{e^x}{e^{2x}+1} dx$

$e^x = t \Rightarrow e^x\ dx = dt \Rightarrow I = \int_1^{\infty} \frac{dt}{t^2+1} = \left[\tan^{-1} t\right]_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

(ii) $I = \int_0^1 \frac{x-1+1}{1+\sqrt{x}} dx = \int_0^1 \frac{(\sqrt{x}-1)\ \ (\sqrt{x}+1)}{(1+\sqrt{x})}\ dx + \int_0^1 \frac{dx}{1+\sqrt{x}}$

Let (माना) $1 + \sqrt{x} = t \Rightarrow \frac{1}{2\sqrt{x}} dx = dt \Rightarrow dx = 2(t-1)\ dt$

$\Rightarrow I_1 = 2\int_1^2 \frac{t-1}{t} dt = -\frac{1}{3} + I_1 = 2 - 2\ell n2 - \frac{1}{3} = \left(\frac{11}{3} - 2\ell n2\right) = \frac{5}{3} - 2\ell n2$. **Ans.**

(iii) $I = \int_0^{\pi/2} \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}$

$\cos x = t \Rightarrow -\sin x\ dx = dt \Rightarrow I = \int_1^0 \frac{t(-dt)}{t^2+3t+2}$ $I = \int_0^1 \frac{t\ \ dt}{(t+1)(t+2)}$

$= \int_0^1 \frac{(t+1)\ \ dt}{(t+1)(t+2)} - \int_0^1 \frac{dt}{(t+1)(t+2)}$

$= 2\left[\ell n(t+2)\right]_0^1 - \int_0^1 \frac{dt}{t+1} = 2[\ell n3 - \ell n2] - [\ell n2] = 2\ell n3 - 3\ell n2 = \ell n\frac{9}{8}$

(iv) $I = \int_0^{\pi/2} \frac{2\sin\theta\cos\theta}{\sin^4\theta + \cos^4\theta} d\theta = \int_0^{\pi/2} \frac{2\tan\theta\sec^2\theta}{\tan^4\theta + 1} d\theta$

Let (माना) tanθ = t ⇒ sec²θ dθ = dt ⇒ $\int_0^\infty \frac{2t\,dt}{t^4+1}$

Let (माना) t² = u ⇒ 2tdt = du ⇒ $\int_0^\infty \frac{du}{u^2+1} = \frac{\pi}{2}$

(v) $I = \int_0^{\pi/4} \frac{\sin x+\cos x}{9+16\sin 2x}dx = \sqrt{2} \int_0^{\pi/4}\frac{\sin\left(x+\frac{\pi}{4}\right)}{9+16\sin 2x}$

using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ (प्रयोग करने पर) $\Rightarrow I = \sqrt{2}\int_0^{\pi/4}\frac{\cos x}{9+16\cos 2x}dx$

$\Rightarrow \quad I = \sqrt{2}\int_0^{\pi/4}\frac{\cos x\ \ dx}{9+16(1-2\sin^2 x)} = \frac{\sqrt{2}}{32}\int_0^{1/\sqrt{2}}\frac{dt}{\frac{25}{32}-t^2}$

$$= \frac{\sqrt{2}}{\sqrt{32}}\,\frac{1}{10}\left[\ell n\left|\frac{\frac{5}{\sqrt{32}}+t}{\frac{5}{\sqrt{32}}-t}\right|\right]_0^{1/\sqrt{2}} = \frac{1}{20}\ell n3$$

A-7 (i) Find the value of a such that $\int_0^a \frac{1}{e^x+4e^{-x}+5}dx = \ell n\sqrt[3]{2}$.

a का मान ज्ञात कीजिए जबकि $\int_0^a \frac{1}{e^x+4e^{-x}+5}dx = \ell n\sqrt[3]{2}$.

Ans. ℓn 11

(ii) Find the value of सरल कीजिए $\int_0^{(\pi/2)^{1/3}} x^5 . \sin x^3\ dx$

Ans. $\frac{1}{3}$

Sol. (i) $I = \int_0^a \frac{e^x}{e^{2x}+5e^x+4}dx$ Let माना eˣ = t ⇒ eˣdx = dt

so इसलिए $I = \int_1^{e^a}\frac{dt}{t^2+5t+4} = \frac{1}{3}\ell n\left|\frac{t+1}{t+4}\right|_1^{e^a}$

Hence अतः $I = \frac{1}{3}\left\{\ell n\left(\frac{e^a+1}{e^a+4}\right)-\ell n\frac{2}{5}\right\} = \frac{1}{3}\ell n2 \Rightarrow \frac{e^a+1}{e^a+4} = \frac{4}{5}$ ⇒ $e^a = 11$

a = ℓn 11

(ii) Let (माना) x³ = t ⇒ 3x² dx = dt

$I = \int_0^{\pi/2}\frac{t}{3}\ \sin t\ dt = 0$

Now apply by parts for solution

अब खण्डशः समाकलन का प्रयोग करने पर

$$I = \left[-\frac{t}{3}\cos t\right]_0^{\pi/2} + \frac{1}{3}\int_0^{\pi/2}\cos t \quad dt \; = \; \frac{1}{3}$$

Section (B) : Definite Integration using Properties

खण्ड (B) : गुणधर्मो की सहायता से निश्चित समाकलन

B-1. Let f(x) = $\ell n\left(\frac{1-\sin x}{1+\sin x}\right)$, then show that $\int_a^b f(x) \; dx \; = \; \int_b^a \ell n \left(\frac{1+\sin x}{1-\sin x}\right) dx$.

माना f(x) = $\ell n\left(\frac{1-\sin x}{1+\sin x}\right)$ हो, तो सिद्ध कीजिए कि $\int_a^b f(x) \; dx \; = \; \int_b^a \ell n \left(\frac{1+\sin x}{1-\sin x}\right) dx$

Sol. $\int_a^b f(x) \; dx = \int_a^b \ell n \left(\frac{1-\sin x}{1+\sin x}\right) \; dx \; = \; \int_a^b -\ell n \left(\frac{1+\sin x}{1-\sin x}\right) \; dx \; = \int_b^a \ell n \left(\frac{1+\sin x}{1-\sin x}\right) \; dx$

B-2. Evaluate :

मान ज्ञात कीजिए–

(i) $\int_0^2 [x^2] \; dx$ (where [.] denotes greatest integer function) जहाँ [.] महत्तम पूर्णांक फलन को प्रदर्शित करता है

(ii) $\int_0^{\pi} \sqrt{1+\sin 2x}\, dx$ (iii) $\int_0^2 f(x)dx$ where जहाँ $f(x) = \begin{cases} 2x+1 & 0 \le x < 1 \\ 3x^2 & 1 \le x \le 2 \end{cases}$

(iv) $\int_0^4 |x^2 - 4x + 3|dx$ (v) $\int_0^{\infty} [\cot^{-1} x]dx$ (where [.] denotes greatest integer function) जहाँ [.] महत्तम पूर्णांक फलन को प्रदर्शित करता है।

(vi) $\int_{-5}^{5} |x+2|\,dx$

(vii) ✍ $\int_{-1}^{1} [\cos^{-1} x]dx$ (where [.] denotes greatest integer function) जहाँ [.] महत्तम पूर्णांक फलन को प्रदर्शित करता है।

Ans.

(i)	$5 - \sqrt{2} - \sqrt{3}$	(ii)	$2\sqrt{2}$	(iii)	9
(iv)	4	(v)	cot 1	(vi)	29
(vii)	cos 1 + cos 2 + cos 3 + 3				

Sol. (i) $\int_0^2 [x^2] \; dx = \int_0^1 0.dx + \int_1^{\sqrt{2}} 1 \; dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 \; dx + \int_{\sqrt{3}}^{2} 3 \; dx$

$= (\sqrt{2}-1) + 2 \; [\sqrt{3} - \sqrt{2}] + 3[2 - \sqrt{3}] \; = 5 - \sqrt{2} - \sqrt{3}$

(ii) $I = \int_0^{\pi} \sqrt{1+\sin 2x} dx \; = \int_0^{\pi} |\sin x + \cos x| dx \; = \sqrt{2}\int_0^{\pi}\left|\sin\left(x+\frac{\pi}{4}\right)\right| dx$

$= \sqrt{2}\left\{\int_0^{3\pi/4} \sin\left(x+\frac{\pi}{4}\right)dx - \int_{3\pi/4}^{\pi} \sin\left(x+\frac{\pi}{4}\right)dx\right\} \; = 2\sqrt{2}$

(iii) $\int_0^2 f(x)dx \; = \; \int_0^1 f(x)dx + \int_1^2 f(x)dx = x^2 + x\Big|_0^1 + x^3\Big|_1^2 \; = 2 + (8 - 1) = 9$

(iv) $\int_0^4 \left|x^2-4x+3\right|dx = \int_0^1 (x^2-4x+3)dx - \int_1^3 (x^2-4x+3)dx + \int_3^4 (x^2-4x+3)dx = 4$

(v) $\int_0^{\infty} [\cot^{-1}x]dx = \int_0^{\cot 1} 1dx + \int_{\cot 1}^{\infty} 0dx = \cot 1$

(vi) $\int_{-5}^{5} |x+2|\,dx = -\int_{-5}^{-2} (x+2)\ \ dx + \int_{-2}^{5} (x+2)\ \ dx = -\left[\frac{x^2}{2}+2x\right]_{-5}^{-2} + \left[\frac{x^2}{2}+2x\right]_{-2}^{5}$

$= -\left[2-4-\left(\frac{25}{2}-10\right)+\left[\frac{25}{2}+10-(2-4)\right]\right] = -\left[-2-\frac{5}{2}\right]+\left[\frac{45}{2}+2\right] = \frac{9}{2}+\frac{49}{2} = \frac{58}{2} = 29$

(vii) $\int_{-1}^{1} [\cos^{-1}x]dx = \int_{-1}^{\cos 3} 3.dx + \int_{\cos 3}^{\cos 2} 2.dx + \int_{\cos 2}^{\cos 1} 1.dx + \int_{\cos 1}^{1} 0.dx$

= 3(cos3+1) + 2 (cos 2 – cos3)+(cos1–cos2) + 0 = cos1 + cos2 + cos3 + 3

B-3. Evaluate :

मान ज्ञात कीजिए–

(i) $\int_{-1}^{1} e^{|x|}\ dx$ (ii) $\int_{-\pi/4}^{\pi/4} |\sin x|\,dx$ (iii) $\int_{-\pi/4}^{\pi/4} \frac{x+\pi/4}{2-\cos 2x}\ dx$

(iv) $\int_{-1}^{1} \sin^5 x \cos^4 x\ \ dx$ (v) $\int_{-\pi/2}^{\pi/2} \frac{g(x)-g(-x)}{f(-x)+f(x)}dx$

Ans. (i) 2e – 2 (ii) $2-\sqrt{2}$ (iii) $\frac{\pi^2}{6\sqrt{3}}$

(iv) 0 (v) 0

Sol. (i) $\int_{-1}^{1} e^{|x|}dx = 2\int_0^1 e^x dx = 2\left[e^x\right]_0^1 = 2[e-1]$

(ii) $\int_{-\pi/4}^{\pi/4} |\sin x|\,dx = 2\int_0^{\pi/4} \sin x dx = -2[\cos x]_0^{\pi/4} = 2-\sqrt{2}$

(iii) $I = \int_{-\pi/4}^{\pi/4} \frac{x+\pi/4}{2-\cos 2x}dx$

$I = 2\int_0^{\pi/4} \frac{\pi/4}{2-\cos 2x}dx$

$I = \frac{\pi}{2}\int_0^{\pi/4} \frac{dx}{3-2\cos^2 x}dx$; $I = \frac{\pi}{2}\int_0^{\pi/4} \frac{\sec^2 x}{3(1+\tan^2 x)-2}\ dx$

$I = \frac{\pi}{2}$; $\int_0^{\pi/4} \frac{\sec^2 x}{3\tan^2 x+1}\ dx$

put tan x = t रखने पर

$I = \frac{\pi}{2}\int_0^1 \frac{dt}{1+3t^2}\ dx$; $I = \frac{\pi}{6}\int_0^1 \frac{dt}{t^2+\left(\frac{1}{\sqrt{3}}\right)^2}\ dx$

$I = \frac{\pi}{6}\ .\ \sqrt{3}\left(\tan^{-1}\left(t\sqrt{3}\right)\right)_0^1$; $I = \frac{\pi}{6}\ .\ \sqrt{3}\ .\ \frac{\pi}{3}$; $I = \frac{\pi^2}{6\sqrt{3}}$

(iv) $\int_{-1}^{1} \sin^5 x \cos^4 x dx = 0$

since $\sin^5x \cos^4x$ is odd function ⇒ चूँकि $\sin^5x \cos^4x$ एक विषम फलन है

(v) given function is odd ⇒ $I = 0$

B-4. Evaluate **[16JM120518]**

मान ज्ञात कीजिए–

(i) $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ (ii) $\int_0^{\pi/2} \frac{e^{\sin x}}{e^{\sin x} + e^{\cos x}} dx$

(iii) $\int_0^{a} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx$ (iv) $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x} dx$

(v) $\int_0^{\pi/2} \frac{\sin x - \cos x}{(\sin x + \cos x)^2} dx$

Ans. (i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{4}$ (iii) $\frac{a}{2}$ (iv) $(a + b)\ \frac{\pi}{4}$ (v) 0

Sol. (i) $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ Applying property (iv) गुणधर्म से

$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x}\ \cdot\ \sqrt{\sin x}}$ ⇒ $2I = \int_0^{\pi/2} dx$ ⇒ $I = \frac{\pi}{4}$

(ii) $I = \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos x\left(\frac{\pi}{2} - x\right)} dx$; $I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x} dx$

$2I = \int_0^{\pi/2} 1.\ dx = \frac{\pi}{2}$ ⇒ $I = \frac{\pi}{4}$

(iii) $I = \int_0^{a} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{a-x}} dx = \int_0^{a} \frac{\sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} dx$; $2I = \int_0^{a} 1.\ dx = a$ ⇒ $I = \frac{a}{2}$

(iv) $I = \int_0^{\pi/2} \frac{a \sin x + b\cos x}{\sin x + \cos x} dx$ Applying property (iv) गुणधर्म से

$I = \int_0^{\pi/2} \frac{a \cos x + b\sin x}{\cos x + \sin x} dx$ ⇒ $2I = \int_0^{\pi/2} \frac{(a+b)\ \sin x + (a+b)\cos x}{\sin x + \cos x} dx$

⇒ $2I = (a + b)\ \frac{\pi}{2}$ ⇒ $I = (a + b)\ \frac{\pi}{4}$

(v) $I = \int_0^{\pi/2} \frac{\sin x - \cos x}{(\sin x + \cos x)^2} dx = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - x\right) - \cos\left(\frac{\pi}{2} - x\right)}{\left(\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)\right)^2} dx$

$= \int_0^{\pi/2} \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx = -I \quad \therefore \quad I = 0$

B-5. Evaluate सरल कीजिए :

(i) $\int_0^{2\pi} \{\sin(\sin x) + \sin(\cos x)\} dx$ (ii) $\int_0^{\pi} \frac{dx}{5 + 4\cos 2x}$

(iii)✍ $\int_0^{\pi/2} (2\ \ell n \sin x - \ell n\ \sin 2x)\, dx$ (iv)✍ $\int_0^{\infty} \ell n\left(x + \frac{1}{x}\right) . \frac{dx}{1 + x^2}$

Ans. (i) 0 (ii) $\frac{\pi}{3}$ (iii) $-\frac{\pi}{2}\ \ell n\ 2$ (iv) $\pi \ell n2$

Sol. (i) $I = \int_0^{2\pi} \{\sin(\sin x)\} dx + \int_0^{2\pi} \sin(\cos x) dx = 0 + 2\int_0^{\pi} \sin(\cos x) dx$ {as sin(sin(2π – x)) = –sin(sinx)}= 0

{as sin(cosx) = –sin(cos(π – x))}

(ii) $\int_0^{\pi} \frac{dx}{5 + 4\cos 2x} = \int_0^{\pi} \frac{\sec^2 x}{9 + \tan^2 x} dx = 2\int_0^{\pi/2} \frac{\sec^2 x}{9 + \tan^2 x} dx = \frac{2}{3} \tan\left(\frac{1}{3}\tan x\right)\Big|_0^{\pi/2} = \frac{\pi}{3}$

(iii) $\int_0^{\pi/2} (2\ \ell n\ \sin x - \ell n\ 2 - \ell n\ \sin x - \ell n\ \cos x)\ dx = -\int_0^{\pi/2} \ell n\ 2\ dx = -\frac{\pi}{2}\ \ell n\ 2$

(iv) Let माना x = tanθ $\therefore I = \int_0^{\pi/2} \ell n(\tan\theta + \cot\theta) d\theta = \int_0^{\pi/2} \ell n\left(\frac{2}{\sin 2\theta}\right) d\theta = \frac{\pi}{2}\ell n 2 - \left(-\frac{\pi}{2}\ell n 2\right) = \pi \ell n 2$

B-6. Evaluate : **[16JM120519]**

(i) $\int_{-1}^{2} \{2x\}\ dx$ (where function {.} denotes fractional part function)

(ii) $\int_0^{10\pi} (|\sin x| + |\cos x|)\ dx$

(iii) $\frac{\int_0^n [x]\ dx}{\int_0^n \{x\}\ dx}$, where [x] and {x} are integral and fractional parts of x and n ∈ N

(iv) $\int_0^{2n\pi} \left(|\sin x| - \left[\left|\frac{\sin x}{2}\right|\right]\right) dx$ (where [ ] denotes the greatest integer function and n ∈ I)

Hindi. मान ज्ञात कीजिए–

(i) $\int_{-1}^{2} \{2x\}\ dx$ (जहाँ फलन {.} भिन्नात्मक भाग फलन को प्रदर्शित करता है)

(ii) $\displaystyle\int_0^{10\pi} (|\sin x| + |\cos x|)\ \ dx$

(iii) $\dfrac{\displaystyle\int_0^n [x]\ \ dx}{\displaystyle\int_0^n \{x\}\ \ dx}$, जहाँ [x] और {x} क्रमशः x के पूर्णांक भाग और भिन्नात्मक भाग को व्यक्त करता है तथा n ∈ N

(iv) $\displaystyle\int_0^{2n\pi} \left(|\sin x| - \left[\left|\frac{\sin x}{2}\right|\right]\right)\ dx$ (जहाँ [ ] महत्तम पूर्णांक फलन है और n ∈ I)

Ans. (i) $\dfrac{3}{2}$ (ii) 40 (iii) n – 1 (iv) 4n

Sol. (i) $\displaystyle\int_{-1}^{2} (2x - [2x]\ \ dx = \int_{-1}^{2} 2x\ \ dx - \int_{-1}^{2} [2x]\ \ dx$

$$= \left[x^2\right]_{-1}^{2} - \left(\int_{-1}^{-1/2} [2x]dx + \int_{-1/2}^{0} [2x]dx + \int_{0}^{1/2} [2x]dx + \int_{1/2}^{1} [2x]dx + \int_{1}^{3/2} [2x]dx + \int_{3/2}^{2} [2x]dx\right)$$

$$= 3 - \left[\left[-2x\right]_{-1}^{-1/2} + \left[-x\right]_{-1/2}^{0} + 0 + \left[x\right]_{1/2}^{1} + \left[2x\right]_{1}^{3/2} + \left[3x\right]_{3/2}^{2}\right]$$

$$= 3 - \left[1 - 2 + 0 - \frac{1}{2} + \frac{1}{2} + 1 + 6 - \frac{9}{2}\right] = \frac{9}{2} - 3 = \frac{3}{2}$$

(ii) $\displaystyle I = \int_0^{20\frac{\pi}{2}} (|\sin x| + |\cos x|)dx$

$$= 20\int_0^{\frac{\pi}{2}} (\sin x + \cos x)dx \qquad \left[\because \ \text{prd of } \ |\sin x| + |\cos x| = \frac{\pi}{2}\right]$$

$\left[\because \ |\sin x| + |\cos x| \right.$ का आवर्तकाल $\left. = \dfrac{\pi}{2}\right]$

= 40

(iii) $\displaystyle\int_0^n [x]\ \ dx = \int_0^1 0.\ \ dx + \int_1^2 1.dx + ... + \int_{n-1}^{n} (n-1)\ \ dx$

$= 1 + 2 + 3 + ...+ (n - 1) = \dfrac{n(n-1)}{2}$ ...... (i) $\quad\therefore\quad \dfrac{\int_0^n [x]dx}{\int_0^n \{x\}dx} = (n-1)$

(iv) $\because \quad 0 \le \left|\dfrac{\sin x}{2}\right| \le \dfrac{1}{2} \qquad \therefore \quad \left[\left|\dfrac{\sin x}{2}\right|\right] = 0$

$$I = \int_0^{2n\pi} |\sin x|dx = 2n\int_0^{\pi} (\sin x)\ \ dx = 4n\int_0^{\pi/2} \cos x\ \ dx = 4n$$

B-7. If f(x) is a function defined ∀ x∈R and f(x) + f(–x) = 0 ∀ x∈ $\left[-\dfrac{T}{2}, \dfrac{T}{2}\right]$ and has period T, then prove that

$\phi(x) = \displaystyle\int_a^x f(t)\ \ dt$ is also periodic with period T.

यदि f(x) $\forall$ x$\in$R के लिए परिभाषित है तथा f(x) + f(–x) = 0 $\forall$ x $\in \left[-\frac{T}{2}, \frac{T}{2}\right]$ है एवं इसका आवर्तनांक T है, तब सिद्ध कीजिए कि $\phi(x) = \int_a^x f(t)\, dt$ भी T आवर्तनांक का आवर्तीफलन है।

Sol. $\phi(x) = \int_a^x f(t)\, dt$

$$\phi(x+T) = \int_a^{x+T} f(t)\, dt = \int_a^x f(t)\, dt + \int_x^{x+T} f(t)\, dt$$

$$= \phi(x) + \int_x^{x+T} f(t)\, dt = \phi(x) + \int_x^{T/2} f(t)\, dt + \int_{T/2}^{x+T} f(t)\, dt$$

$$= \phi(x) + \int_x^{T/2} f(t)\, dt + \int_{-T/2}^{x} f(t+T)\, dt$$

(by using shifting property) (स्थानांतरण गुणधर्म से)

$$= \phi(x) + \int_{-T/2}^{x} f(t)\, dt + \int_x^{T/2} f(t)\, dt = \phi(x) + \int_{-T/2}^{T/2} f(t)\, dt$$

$$= \phi(x) + 0 = \phi(x) \left[\because \text{ f(x) is odd on } \left[-\frac{T}{2}, \frac{T}{2}\right]\right] \left[\because \left[-\frac{T}{2}, \frac{T}{2}\right] \text{ में f(x) विषम है}\right]$$

$\therefore$ $\phi(x + T) = \phi(x)$

Section (C) : Leibnitz formula and Wallis' formula
खण्ड (C) : लेबनीज सूत्र एवं वॉलीस सूत्र (Wallis' Formula)

C-1. (i) If f(x) = $5^{g(x)}$ and $g(x) = \int_2^{x^2} \frac{t}{\ell n\ (1+t^2)}\, dt$, then find the value of $f'(\sqrt{2})$.

यदि f(x) = $5^{g(x)}$ और $g(x) = \int_2^{x^2} \frac{t}{\ell n\ (1+t^2)}\, dt$ हो, तो $f'(\sqrt{2})$ का मान ज्ञात कीजिए।

(ii) The value of $\lim\limits_{x\to 0} \dfrac{\frac{d}{dx}\int_0^{x^3}\sqrt{\cos t}\ dt}{1-\sqrt{\cos x}}$

$\lim\limits_{x\to 0} \dfrac{\frac{d}{dx}\int_0^{x^3}\sqrt{\cos t}\ dt}{1-\sqrt{\cos x}}$ का मान है।

(iii) Find the slope of the tangent to the curve $y = \int_x^{x^2} \cos^{-1}t^2\, dt$ at $x = \frac{1}{\sqrt[4]{2}}$

वक्र $y = \int_x^{x^2} \cos^{-1}t^2\, dt$ की $x = \frac{1}{\sqrt[4]{2}}$ पर स्पर्श रेखा की प्रवणता है।

Ans. (i) $4\sqrt{2}$ (ii) 12 (iii) $\left(\frac{\sqrt[4]{8}}{3} - \frac{1}{4}\right)\pi$

Sol. (i) $f'(x) = 5^{g(x)}\ \ell n5\ g'(x)$

Now अब $g'(x) = \frac{x^2 \cdot 2x}{\ell n\ (1+x^4)}$ $\therefore$ $f'(x) = 5^{g(x)}\ \ell n5\ \frac{2\ x^3}{\ell n\ (1+x^4)}$

$$\Rightarrow \quad f'(\sqrt{2}) = 5^{g(\sqrt{2})}\ \ell n5\ \frac{2(\sqrt{2})^3}{\ell n(1+(\sqrt{2})^4)} = 4\sqrt{2}$$ { since चूँकि $g(\sqrt{2}) = 0$}

(ii) $$\lim_{x\to 0} \frac{\sqrt{\cos x^3}\ .\ 3x^2}{1-\sqrt{\cos x}} = \lim_{x\to 0} \frac{3x^2}{1-\cos x}\left(1+\sqrt{\cos x}\right) = 3.2.2 = 12$$

(iii) given curve is दिया है $y = \int_x^{x^2} \cos^{-1} t^2\ dt$; $\frac{dy}{dx} = \frac{d}{dx}\int_x^{x^2} \cos^{-1} t^2\ dt$

using Leibnitz theorem, लेबनीज प्रमेय के प्रयोग से–

$$\frac{dy}{dx} = 2x\ \cos^{-1} x^4 - \cos^{-1} x^2$$

$$\left(\frac{dy}{dx}\right)_{x = \frac{1}{\sqrt[4]{2}}} = \frac{2}{2^{1/4}} \cos^{-1}\frac{1}{2} - \cos^{-1}\frac{1}{\sqrt{2}} = 2^{3/4}\frac{\pi}{3} - \frac{\pi}{4} = \left(\frac{\sqrt[4]{8}}{3} - \frac{1}{4}\right)\pi$$

C-2. (i) If $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\ \ dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\ \ dt$, then prove that $f'(x) = 0\ \ \forall\ x \in R$.

यदि $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\ \ dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\ \ dt$ हो, तो सिद्ध कीजिए कि $f'(x) = 0\ \ \forall\ x \in R$

[16JM120520]

(ii) Find the value of x for which function $f(x) = \int_{-1}^{x} t(e^t - 1)\ (t - 1)\ (t - 2)^3\ (t - 3)^5\ dt$ has a local minimum

x का मान ज्ञात कीजिए जिसके लिए फलन $f(x) = \int_{-1}^{x} t\ (e^t - 1)\ (t - 1)\ (t - 2)^3\ (t - 3)^5$ स्थानीय निम्निष्ठ है।

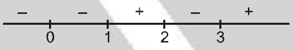
Ans. (ii) 1, 3

Sol. (i) $f'(x) = \sin^{-1}\sqrt{\sin^2 x}\ .\ 2 \sin x \cos x - \cos^{-1}\sqrt{\cos^2 x}\ .\ 2 \cos x \sin x$

$= x\ .\ \sin 2x - x \sin 2x = 0$

(ii) $f(x) = \int_{-1}^{x} t(e^t-1)(t-1)(t-2)^3(t-3)^5\ \ dt$; $\frac{dy}{dx} = f'(x) = x.(e^x-1)(x - 1)\ (x-2)^3\ (x-3)^5$

change of sign चिन्ह परिवर्तन से



Points of minima x = 1, 3निम्निष्ठ के बिन्दु x = 1, 3

C-3. If $y = \int_1^x x\sqrt{\ell nt}\,dt$

then find the value of $\frac{d^2y}{dx^2}$ at x = e

Hindi. यदि $y = \int_1^x x\sqrt{\ell nt}\,dt$ हो तो $\frac{d^2y}{dx^2}$ का x = e पर मान ज्ञात कीजिए।

Ans. 5/2

Sol. $$\frac{dy}{dx} = x\sqrt{\ell nx} + \int_1^x \sqrt{\ell nt}\ dt \quad \Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{2\sqrt{\ell nx}} + \sqrt{\ell nx} + \sqrt{\ell nx}$$

$$\Rightarrow \left.\frac{d^2y}{dx^2}\right|_{x=e} = \frac{1}{2}+1+1=\frac{5}{2}$$

C-4. $\lim\limits_{n\to\infty}\frac{\int\limits_{1/(n+1)}^{1/n}\tan^{-1}(nx)dx}{\int\limits_{1/(n+1)}^{1/n}\sin^{-1}(nx)dx}$ is equal to बराबर है

Ans. $\frac{1}{2}$

Sol. Put nx = t we get

nx = t रखने पर

$$\lim_{n\to\infty}\frac{\frac{1}{n}\int\limits_{n/n+1}^{1}\tan^{-1}(t)dt}{\frac{1}{n}\int\limits_{n/n+1}^{1}\sin^{-1}(t)dt} = \lim_{n\to\infty}\frac{\tan^{-1}\left(\frac{n}{n+1}\right)}{\sin^{-1}\left(\frac{n}{n+1}\right)} = \frac{\pi/4}{\pi/2} = \frac{1}{2}$$

C-5. Let f be a differentiable function on R and satisfying the integral equation

$x\int\limits_0^x f(t)dt - \int\limits_0^x tf(x-t)dt = e^x - 1 \quad \forall\ x \in R$, then f(1) equals to

माना f, R पर अवकलनीय फलन है तथा समीकरण $x\int\limits_0^x f(t)dt - \int\limits_0^x tf(x-t)dt = e^x - 1 \quad \forall\ x \in R$

को संतुष्ट करता है तब f(1) बराबर है–

Ans. e

Sol. $x\int\limits_0^x f(t)dt - x\int\limits_0^x (x-t)f(t)dt = e^x - 1 \Rightarrow \int\limits_0^x tf(t)dt = e^x - 1 \Rightarrow xf(x) = e^x \Rightarrow f(1) = e$

C-6. Evaluate : **[16JM120521]**

मान ज्ञात कीजिए–

(i) $\int\limits_{-\pi/2}^{\pi/2}\sin^2 x\cos^2 x(\sin x+\cos x)\ \ dx$ **Ans.** $\frac{4}{15}$

(ii) $\int\limits_0^{\pi} x\sin^5 x\ \ dx$ **Ans.** $\frac{8\pi}{15}$

(iii) $\int\limits_0^{2} x^{3/2}\sqrt{2-x}\ \ dx$ **Ans.** $\frac{\pi}{2}$

(iv) $\int\limits_0^{2\pi} x\ (\sin^2 x\cos^2 x)\ dx$ **Ans.** $\frac{\pi^2}{4}$

Sol. (i) $I = \int\limits_{-\pi/2}^{\pi/2}\sin^2 x\cos^2 x\ \ (\sin x+\cos x)\ \ dx$; $I = \int\limits_{-\pi/2}^{\pi/2}\sin^2 x\ \ \cos^2 x\ \ [-\sin x+\cos x]\ \ dx$

using $\int\limits_a^b f(x)\ \ dx = \int\limits_a^b f(a+b-x)\ \ dx$ (का प्रयोग करने पर)

$$\Rightarrow \quad 2I = \int\limits_{-\pi/2}^{\pi/2}\sin^2 x\cos^3 x\ \ dx = \frac{2(2-1).(3-1)}{(2+3)\ \ (3).1}.1 = \frac{4}{15}$$

(ii) $I = \int_0^{\pi} x \quad \sin^5 x \quad dx$

$I = \int_0^{\pi} (\pi - x) \quad \sin^5 x \quad dx \Rightarrow \qquad 2I = \pi \quad \int_0^{\pi} \sin^5 x \quad dx \Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \sin^5 x \quad dx$

By using property (v) गुणधर्म से

$= 2\frac{\pi}{2} \int_0^{\pi/2} \sin^5 x \, dx$

By applying Wallis formulae वॉली सूत्र से

$= \pi.\frac{(5-1)(5-3)}{(5-2)(5-4)} = \frac{8\pi}{15}$

(iii) $I = \int_0^2 x^{3/2} \quad \sqrt{2-x} \quad dx$

Let (माना) $x = 2\sin^2\theta \qquad \Rightarrow \qquad dx = 4\sin\theta \cos\theta \, d\theta$

$I = \int_0^{\pi/2} 2^{3/2} \cdot \sin^3\theta\sqrt{2}\cos\theta \quad . \quad 4\sin\theta \quad \cos\theta \quad d\theta = 16 \int_0^{\pi/2} \sin^4\theta \quad \cos^2\theta \quad d\theta$

$= 16 \frac{(4-1)(4-3) \quad \cdot \quad (2-1)}{6.4.2}$ [Using walli's formula] [वाली का सूत्र प्रयोग करने पर]

$= \frac{3\pi}{6} = \frac{\pi}{2}$

(iv) $I = \int_0^{2\pi} x \quad \sin^2 x \cos^2 x \quad dx \quad = \int_0^{2\pi} (2\pi - x) \quad \sin^2 x \cos^2 x \quad dx$

$I = \pi \int_0^{2\pi} \sin^2 x \cos^2 x \quad dx \quad = \frac{\pi}{4} \int_0^{2\pi} 4\sin^2 x \quad \cos^2 x \quad dx \quad = \frac{\pi}{4} \int_0^{2\pi} \sin^2 2x \quad dx$

$= \frac{\pi}{8} \int_0^{2\pi} (1 - \cos 4x) \quad dx \quad = \frac{\pi}{8}\left(x - \frac{\sin 4x}{4}\right)\Big|_0^{2\pi} \quad = \frac{\pi}{8} \quad (2\pi) = \frac{\pi^2}{4}$

SECTION (D) : ESTIMATION & MEAN VALUE THEOREM
SECTION (D) : अनुमानित मान तथा माध्य मान प्रमेय

D-1. Prove the following inequalities : –
निम्न असमिकाओं को सिद्ध कीजिए–

(i) $\frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx < \frac{\sqrt{2}}{6}$ (ii) $4 \le \int_1^3 \sqrt{(3+x^3)}\, dx \le 2\sqrt{30}$

Sol. (i) $\because \quad \frac{\sin x}{x}$ is monotonic decreasing $\quad \therefore \quad \frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} \quad dx < \frac{\sqrt{2}}{6}$

(ii) Let $f(x) = \sqrt{3+x^3}$

$f'(x) = \frac{3x^2}{2\sqrt{3+x^3}} \quad > 0 \ \forall \ x \in (1, 3)$

⇒ f is strictly increasing in (1, 3) ∵ m = least value = f(1) = $\sqrt{3+1} = 2$

M = greatest value = f(3) = $\sqrt{30}$ ∵ $m(b-a) \le \int_a^b f(x)\ dx \le M(b-a)$

⇒ $2(2) \le \int_1^3 \sqrt{3+x^3}\ dx \le \sqrt{30} \times 2$ ⇒ $4 \le \int_1^3 \sqrt{3+x^3}\ dx \le 2\sqrt{30}$

Hindi (i) ∵ $\frac{\sin x}{x}$ एकदिष्ट ह्रसमान है। ∴ $\frac{\sqrt{3}}{8} < \int_{\pi/4}^{\pi/3} \frac{\sin x}{x}\ dx < \frac{\sqrt{2}}{6}$

(ii) माना f(x) = $\sqrt{3+x^3}$

$f'(x) = \frac{3x^2}{2\sqrt{3+x^3}} > 0\ \forall\ x \in (1, 3)$

(1, 3) में f एकदिष्ट वर्द्धमान है। ∵ m = निम्नतम मान = f(1) = $\sqrt{3+1} = 2$

M = अधिकतम मान = f(3) = $\sqrt{30}$ ∵ $m(b-a) \le \int_a^b f(x)\ dx \le M(b-a)$

⇒ $2(2) \le \int_1^3 \sqrt{3+x^3}\ dx \le \sqrt{30} \times 2$ ⇒ $4 \le \int_1^3 \sqrt{3+x^3}\ dx \le 2\sqrt{30}$

D-2. Show that दर्शाइये कि **[16JM120522]**

(i) $\frac{1}{10\sqrt{2}} < \int_0^1 \frac{x^9}{\sqrt{1+x}} dx < \frac{1}{10}$ (ii) $\frac{1}{2}\ell n2 < \int_0^1 \frac{\tan x}{1+x^2} dx < \frac{\pi}{2}$

Sol. (i) $\frac{x^9}{\sqrt{2}} \le \frac{x^9}{\sqrt{1+x}} \le x^9$ $\forall\ x \in [0, 1]$

⇒ $\int_0^1 \frac{x^9}{\sqrt{2}} dx < \int_0^1 \frac{x^9}{\sqrt{1+x}} dx < \int_0^1 x^9 dx$ ⇒ $\frac{1}{10\sqrt{2}} < \int_0^1 \frac{x^9}{\sqrt{1+x}} dx < \frac{1}{10}$

(ii) x < tan x < tan 1 < 2 $\forall\ x \in (0, 1)$

⇒ $\frac{1}{2}\ell n2 < \int_0^1 \frac{\tan x}{1+x^2} dx < 2\int_0^1 \frac{dx}{1+x^2}$ ⇒ $\frac{1}{2}\ell n2 < \int_0^1 \frac{\tan x}{1+x^2} dx < \frac{\pi}{2}$

D-3. **(i)** Show that $\int_0^2 \sin x.\cos\sqrt{x}\ dx = 2\sin c.\cos\sqrt{c}$ for some c∈(0, 2)

दर्शाइये कि $\int_0^2 \sin x.\cos\sqrt{x}\ dx = 2\sin c.\cos\sqrt{c}$ किसी c ∈ (0, 2) के लिए

(ii). f(x) is a continuous function x ∈ R, then show that $\int_1^4 f(x)dx = 2\alpha f(\alpha^2)$ some α ∈ (1, 2)

Hindi. $f(x)$, $x \in R$ के लिए सतत फलन है तब प्रदर्शित कीजिए $\int_1^4 f(x)dx = 2\alpha f(\alpha^2)$ किसी $\alpha \in (1, 2)$ के लिए

Sol. (i) By mean value theorem for a cont. function $\int_a^b f(x)dx = f(c)(b - a)$ for some $c \in (a, b)$

Hence $\int_0^2 \sin x \cos \sqrt{x} dx = \sin c \cos \sqrt{c} \; (2 - 0)$ for some $c \in (0, 2)$.

(ii) $I = \int_1^4 f(x)dx$, let $x = t^2$; $I = \int_1^2 f(t^2).2tdt = 2\alpha f(\alpha^2)(2 - 1)$ for some $\alpha \in (1, 2)$

Hindi (i) माध्य मान प्रमेय से सतत फलन के लिए $\int_a^b f(x)dx = f(c)(b - a)$ किसी $c \in (a, b)$ के लिए

अतः $\int_0^2 \sin x \cos \sqrt{x} dx = \sin c \cos \sqrt{c} \; (2 - 0)$ अतः $c \in (0, 2)$.

(ii) $I = \int_1^4 f(x)dx$, माना $x = t^2$; $I = \int_1^2 f(t^2).2tdt = 2\alpha f(\alpha^2)(2 - 1)$ किसी $\alpha \in (1, 2)$ के लिए

Section (E) : Integration as a limit of sum and reduction formula
खण्ड (E) : सीमाओं के योग से समाकलन तथा समानयन सूत्र

E-1. Evaluate :
मान ज्ञात कीजिए–

(i) $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}}$ **Ans.** $\frac{\pi}{2}$

(ii)✍ $\lim\limits_{n\to\infty} \frac{3}{n}\left[1 + \sqrt{\frac{n}{n+3}} + \sqrt{\frac{n}{n+6}} + \sqrt{\frac{n}{n+9}} + ... + \sqrt{\frac{n}{n+3(n-1)}}\right]$ **Ans.** 2

(iii) $\lim\limits_{n\to\infty} \frac{1}{n^4}\left(\sum\limits_{r=1}^{2n}(3nr^2 + 2n^2r)\right)$ **Ans.** 12

Sol. (i) $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \frac{1}{\sqrt{n^2 - r^2}} = \lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \frac{1}{n} \frac{1}{\sqrt{1 - \left(\frac{r}{n}\right)^2}} = \int_0^1 \frac{1}{\sqrt{1 - x^2}} dx = \left[\sin^{-1} x\right]_0^1 = \frac{\pi}{2}$

(ii) $\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \frac{3}{n}\sqrt{\frac{n}{n+3r}} = \lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \frac{3}{n}\sqrt{\frac{1}{1+3r/n}} = \int_0^1 \sqrt{\frac{1}{1+3x}} dx = \int_0^1 (1+3x)^{-1/2} dx = 2$

(iii) $\lim\limits_{n\to\infty} \frac{1}{n^4}\left(\sum\limits_{r=1}^{2n}(3nr^2 + 2n^2r)\right) = \lim\limits_{n\to\infty} \frac{1}{n}\left(\sum\limits_{r=1}^{2n} 3\left(\frac{r}{n}\right)^2 + 2\left(\frac{r}{n}\right)\right) = \int_0^2 (3x^2 + 2x)dx = x^3 + x^2\Big|_0^2 = 12$

E-2. (i) If $I_n = \int_0^{\pi/4} \tan^n x \; dx$, then show that $I_n + I_{n-2} = \frac{1}{n-1}$ **[16JM120523]**

यदि $I_n = \int_0^{\pi/4} \tan^n x \;\; dx$ हो, तब प्रदर्शित कीजिए कि $I_n + I_{n-2} = \frac{1}{n-1}$

(ii)✍ $I_n = \int_0^{\pi/2} (\sin x)^n dx,\ n \in N$. Show that दर्शाइये कि $I_n = \frac{n-1}{n} I_{n-2} \quad \forall n \geq 2$

Sol. **(i)** $I_n = \int_0^{\pi/4} \tan^n x \;\; dx \quad \Rightarrow \quad I_{n-2} = \int_0^{\pi/4} \tan^{n-2} x \;\; dx$

$$I_n + I_{n-2} = \int_0^{\pi/4} (\tan^n x + \tan^{n-2} x) \;\; dx = \int_0^{\pi/4} \tan^{n-2} x.\;\; \sec^2 x \;\; dx = \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\pi/4} = \frac{1}{n-1}$$

(ii) $I_n = \int_0^{\pi/2} (\sin x)^{n-1} \sin x \, dx = (\sin x)^{n-1}(\cos x) \Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} (\sin x)^{n-2} \cos^2 x dx$

$$\Rightarrow \quad I_n = (n-1) \int_0^{\pi/2} (\sin x)^{n-2} (1 - \sin^2 x) dx \Rightarrow I_n = (n-1) I_{n-2} - (n-1) I_n \Rightarrow I_n = \frac{n-1}{n} I_{n-2}$$

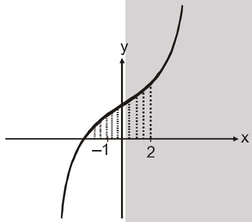
Section (F) : Area Under Curve

खण्ड (F) : वक्र से परिबद्ध क्षेत्रफल

F-1. Find the area enclosed between the curve $y = x^3 + 3$, $y = 0$, $x = -1$, $x = 2$.
वक्र $y = x^3 + 3$, $y = 0$, $x = -1$ तथा $x = 2$ से परिबद्ध क्षेत्रफल ज्ञात कीजिए।

Ans. $\frac{51}{4}$ sq. unit (वर्ग इकाई)

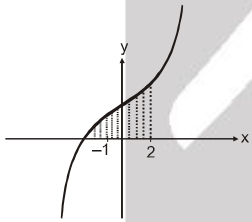
Sol.



Graph of y=x³+3

$$\text{Area} = \int_{-1}^{2} (x^3 + 3) dx = \left(\frac{16}{4} + 3.2\right) - \left(\frac{1}{4} + 3(-1)\right) = \frac{51}{4}$$

Hindi



Graph of y=x³+3

$$\text{क्षेत्रफल} = \int_{-1}^{2} (x^3 + 3) dx = \left(\frac{16}{4} + 3.2\right) - \left(\frac{1}{4} + 3(-1)\right) = \frac{51}{4}$$

F-2. (i) Find the area bounded by $x^2 + y^2 - 2x = 0$ and $y = \sin \frac{\pi x}{2}$ in the upper half of the circle.

(ii) Find the area bounded by the curve $y = 2x^4 - x^2$, x-axis and the two ordinates corresponding to the minima of the function. **[16JM120524]**

(iii) Find area of the curve $y^2 = (7 - x)(5 + x)$ above x–axis and between the ordinates $x = -5$ and $x = 1$.

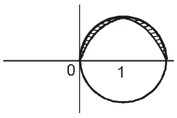
Hindi. (i) वक्र $x^2 + y^2 - 2x = 0$ और $y = \sin\frac{\pi x}{2}$ का क्षेत्रफल वृत्त के ऊपरी भाग में ज्ञात कीजिए।

(ii) वक्र $y = 2x^4 - x^2$, x-अक्ष तथा फलन के निम्निष्ठ के संगत दो कोटियों से परिबद्ध क्षेत्रफल है।

(iii) वक्र $y^2 = (7 - x)(5 + x)$ का क्षेत्रफल ज्ञात कीजिए जो $x = -5$ और $x = 1$ के मध्य x-अक्ष से ऊपर है।

Ans. (i) $\frac{\pi}{2} - \frac{4}{\pi}$ (ii) $\frac{7}{120}$ (iii) 9π

Sol. **(i)** $(x - 1)^2 + y^2 = 1$



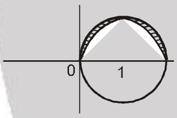
Figure

Area of circle is $\pi r^2 = \pi$

$$\text{Required area} = \frac{\pi}{2} - \int_0^2 \sin\frac{\pi x}{2}\,dx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}$$

(i) $(x - 1)^2 + y^2 = 1$

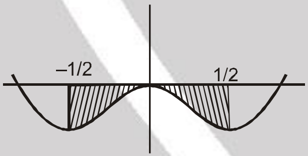


Figure

वृत्त का क्षेत्रफल $\pi r^2 = \pi$ है। चूंकि $r = 1$

$$\text{अभीष्ट क्षेत्रफल} = \frac{\pi}{2} - \int_0^2 \sin\frac{\pi x}{2}\,dx$$

$$= \frac{\pi}{2} - \frac{4}{\pi}$$



Figure

(ii) $\frac{dy}{dx} = 8x^3 - 2x.\ \frac{dy}{dx} = 0$

⇒ $(4x^2 - 1)\,x = 0$

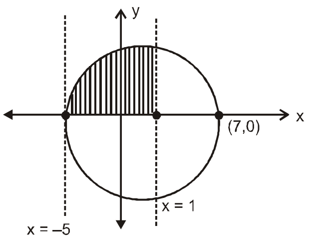
⇒ $x = -\frac{1}{2}, 0, \frac{1}{2}$

Required area अभीष्ट क्षेत्रफल $= -2\int_0^{1/2} (2x^4 - x^2)\ dx = \frac{7}{120}$

(iii) $y^2 = (7 - x)(5 + x)$ at $y = 0$, $x = 7, -5$

$(x - 1)^2 + y^2 = (6)^2$ centre $(1,0)$

From the figure it is clear that area is



$$\int_{-5}^{1} y \; dx = \int_{-5}^{1} \sqrt{(6)^2-(x-1)^2} \; dx$$

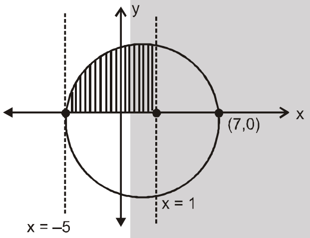
$$= \left[\frac{x-1}{2}\sqrt{6^2-(x-1)^2}+\frac{36}{2}\sin^{-1}\left(\frac{x-1}{6}\right)\right]_{-5}^{1}$$

$$= \left[(0+0)-\left(0+18\sin^{-1}(-1)\right)\right] = 9\pi$$

Hindi. $y^2 = (7-x)(5+x)$, $y = 0$ पर , $x = 7, -5$

$(x-1)^2 + y^2 = (6)^2$, केन्द्र (1,0)

चित्र से स्पष्ट है कि



क्षेत्रफल $= \int_{-5}^{1} y \; dx = \int_{-5}^{1} \sqrt{(6)^2-(x-1)^2} \; dx$

$$= \left[\frac{x-1}{2}\sqrt{6^2-(x-1)^2}+\frac{36}{2}\sin^{-1}\left(\frac{x-1}{6}\right)\right]_{-5}^{1}$$

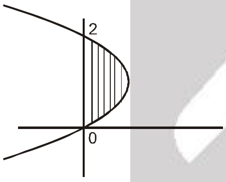
$$= \left[(0+0)-\left(0+18\sin^{-1}(-1)\right)\right] = 9\pi$$

F-3. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.

वक्र $y^2 = 2y - x$ तथा y-अक्ष से परिबद्ध क्षेत्र का क्षेत्रफल कीजिए।

Ans. 4/3 sq. units (वर्ग इकाई)

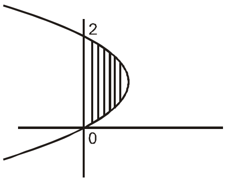
Sol.



Graph of $x = 2y - y^3$

$$\text{Area} = \int_{0}^{2}(2y-y^3)dy = \frac{4}{3}$$

Hindi



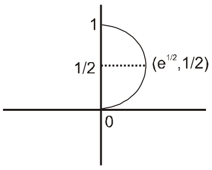
$x = 2y - y^3$ का आरेख

$$\text{क्षेत्रफल} = \int_{0}^{2}(2y-y^3)dy = \frac{4}{3}$$

F-4. Find the area bounded by the y-axis and the curve $x = e^y \sin \pi y$, $y = 0$, $y = 1$. **[16JM120525]**
y-अक्ष, वक्र $x = e^y \sin \pi y$, $y = 0$ तथा $y = 1$ से परिबद्ध क्षेत्र का क्षेत्रफल ज्ञात कीजिए।

Ans. $\frac{(e+1)\ \ \pi}{1+\pi^2}$

Sol. Area क्षेत्रफल $= \int_0^1 e^y \sin(\pi y)\ \ dy$



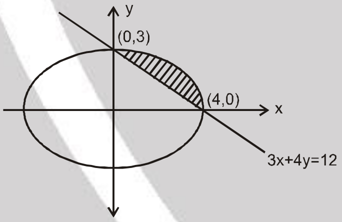
Figure

$= \left.\frac{e^y}{1+\pi^2}(\sin \ \ \pi y - \pi \ \ \cos \pi y)\right]_0^1$

$= \frac{(e+1)\pi}{1+\pi^2}$

F-5. **(i)** Find the area bounded in the first quadrant between the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the line $3x + 4y = 12$
दीर्घवृत्त $\frac{x^2}{16} + \frac{y^2}{9} = 1$ और रेखा $3x + 4y = 12$ के मध्य प्रथम चतुर्थांश में परिबद्ध क्षेत्रफल ज्ञात कीजिए।
(ii) Find the area of the region bounded by $y = \{x\}$ and $2x - 1 = 0$, $y = 0$, ({ } stands for fraction part)
वक्रों $y = \{x\}$, $2x - 1 = 0$ एवं $y = 0$ से परिबद्ध क्षेत्रफल ज्ञात कीजिए। (जहाँ { } भिन्नात्मक भागफल है)

Ans. (i) $3\ (\pi - 2)$ (ii) $\frac{1}{8}$



Sol. **(i)** Area क्षेत्रफल = $= \int_0^4 \left(3\sqrt{1-\frac{x^2}{16}} - \frac{1}{4}(12-3x)\right)\ dx$

$= \frac{3}{4}\left[\frac{x}{4}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\left(\frac{x}{4}\right)\right]_0^4 - \frac{1}{4}\left[12x - \frac{3x^2}{2}\right]_0^4$

$= \frac{3}{4}\left[(0+4\pi)-(0+0)\right] - \frac{1}{4}[48-24]$

$= 3\pi - 6 = 3(\pi - 2)$

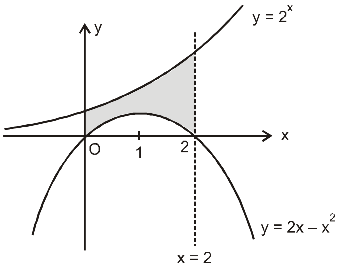
(ii) $\therefore$ $0 < x < \frac{1}{2}$ $\{x\} = x$

$A = \int_0^{1/2} x\ \ .dx = \left(\frac{x^2}{2}\right)_0^{1/2} = \frac{1}{8}$

F-6. Compute the area of the figure bounded by straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$ and $y = 2x - x^2$ **[16JM120526]**
सरल रेखाओं $x = 0$, $x = 2$ और वक्रों $y = 2^x$ तथा $y = 2x - x^2$ से परिबद्ध क्षेत्र का क्षेत्रफल ज्ञात कीजिए।

Ans. $\left(\frac{3}{\log_e 2}-\frac{4}{3}\right)$ sq. units (वर्ग इकाई)

Sol. Area क्षेत्रफल = $\int_0^2 (2^x - 2x + x^2)\ \ dx$



$$= \left(\frac{2^x}{\ell n2} - x^2 + \frac{x^3}{3}\right)_0^2 = \frac{4}{\ell n2} - 4 + \frac{8}{3} - \frac{1}{\ell n2} = \frac{3}{\ell n2} - \frac{4}{3}$$

F-7. Let f(x) = $\sqrt{\tan x}$. Show that area bounded by y = f(x), y = f(c), x = 0 and x = a, 0 < c < a < $\frac{\pi}{2}$ is minimum when c = $\frac{a}{2}$

माना कि f(x) = $\sqrt{\tan x}$ प्रदर्शित कीजिए कि y = f(x), y = f(c), x = 0 तथा x = a, 0 < c < a < $\frac{\pi}{2}$ से परिबद्ध क्षेत्रफल न्यूनतम होगा जबकि $c = \frac{a}{2}$

Sol. Graph of f (x) = $\sqrt{\tan x}$

Let A be area.

$$A = \int_0^c (f(c) - f(x))dx + \int_c^a (f(x) - f(c))dx$$

$$\frac{dA}{dc} = (2c - a)\ \frac{\sec^2 c}{2\sqrt{\tan c}}$$

$$\frac{dA}{dc} = 0 \Rightarrow \quad c = \frac{a}{2}$$

At $c = \frac{a}{2}$, $\frac{dA}{dc}$ changes sign from negative to positive. Hence A is minimum when c = $\frac{a}{2}$

Hindi f (x) = $\sqrt{\tan x}$ का आरेख है

माना A क्षेत्रफल है .

$$A = \int_0^c (f(c) - f(x))dx + \int_c^a (f(x) - f(c))dx$$

$$\frac{dA}{dc} = (2c-a)\ \frac{\sec^2 c}{2\sqrt{\tan c}}$$

$$\frac{dA}{dc} = 0 \Rightarrow \qquad c = \frac{a}{2}$$

$c = \frac{a}{2}$ पर $\frac{dA}{dc}$ का चिन्ह ऋणात्मक से धनात्मक बदलता है अतः A पर निम्निष्ठ है जबकि $c = \frac{a}{2}$

F-8. Find the area included between the parabolas $y^2 = x$ and $x = 3 - 2y^2$. **[16JM120527]**

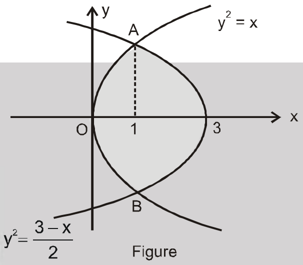
परवलयों $y^2 = x$ और $x = 3 - 2y^2$ के मध्य क्षेत्र का क्षेत्रफल ज्ञात कीजिए।

Ans. 4 sq. units. (वर्ग इकाई)

Sol. $y^2 = x$

$y^2 = \frac{3-x}{2}$ for A and B A तथा B के लिए

$x = \frac{3-x}{2}$ $x = 1,\ y = \pm\ 1$



Figure

A(1, 1) and B(1, –1)

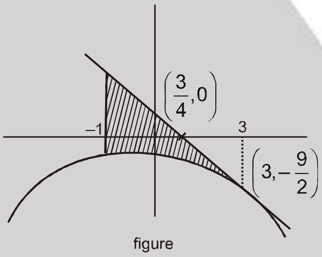
$$A = \int_{-1}^{1}(3-2y^2-y^2)\ \ dy = 2\left(3y - \frac{3y^3}{3}\right)_0^1 = 4$$

F-9. A tangent is drawn to the curve $x^2 + 2x - 4ky + 3 = 0$ at a point whose abscissa is 3. This tangent is perpendicular to $x + 3 = 2y$. Find the area bounded by the curve, this tangent and ordinate $x = -1$

वक्र $x^2 + 2x - 4ky + 3 = 0$ पर स्थित एक बिन्दु, जिसका कि भुज 3 है, पर स्पर्श रेखा खींची जाती है। यदि यह स्पर्श रेखा, रेखा $x + 3 = 2y$ के लम्बवत् है तो वक्र, स्पर्श रेखा तथा कोटि $x = -1$ से परिबद्ध क्षेत्र का क्षेत्रफल ज्ञात कीजिए।

Ans. $\frac{16}{3}$ sq. units. (वर्ग इकाई)

Sol. $x^2 + 2x - 4ky + 3 = 0$



figure

$2x + 2 - 4k\ \frac{dy}{dx} = 0$

put x = 3, $\frac{dy}{dx} = -2$

$6 + 2 + 8k = 0$

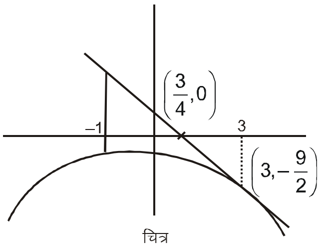
$k = -1$

$y = -\ \frac{1}{4}\ (x^2 + 2x + 3)$

Tangent is $4x + 2y - 3 = 0$

$$\text{Area} = \int_{-1}^{3}\left(\left(\frac{3-4x}{2}\right)-\left(-\frac{1}{4}(x^2+2x+3)\right)\right)dx = \frac{16}{3}$$

Hindi. $x^2 + 2x - 4ky + 3 = 0$



चित्र

$2x + 2 - 4k\frac{dy}{dx} = 0$

x = 3 रखने पर, $\frac{dy}{dx} = -2$

$6 + 2 + 8k = 0$
$k = -1$

$y = -\frac{1}{4}(x^2 + 2x + 3)$

स्पर्श रेखा $4x + 2y - 3 = 0$

$$\text{क्षेत्रफल} = \int_{-1}^{3}\left(\left(\frac{3-4x}{2}\right)-\left(-\frac{1}{4}(x^2+2x+3)\right)\right)dx = \frac{16}{3}$$

F-10. (i) Draw graph of $y = (\tan x)^n$, $n \in \left[0, \frac{\pi}{4}\right]$ N, $x \in \left[0, \frac{\pi}{4}\right]$. Hence show

$0 < (\tan x)^{n+1} < (\tan x)^n$, $x \in \left(0, \frac{\pi}{4}\right)$ **[16JM120528]**

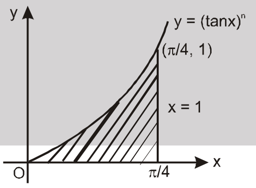
$y = (\tan x)^n$, $n \in \left[0, \frac{\pi}{4}\right]$ N, $x \in \left[0, \frac{\pi}{4}\right]$ का आरेख बनाइये तथा इस प्रकार प्रदर्शित कीजिए कि

$0 < (\tan x)^{n+1} < (\tan x)^n$, $x \in \left(0, \frac{\pi}{4}\right)$

(ii) Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \pi/4$. Prove that for $n > 2$, $A_n + A_{n-2} = 1/(n-1)$ and deduce that $1/(2n+2) < A_n < 1/(2n-2)$.

मानाकि A_n वक्र $y = (\tan x)^n$ तथा रेखाओं $x = 0$, $y = 0$ एवं $x = \pi/4$ से परिबद्ध क्षेत्रफल है। सिद्ध कीजिए कि $n > 2$ के लिए, $A_n + A_{n-2} = 1/(n-1)$ है और निगमन कीजिए कि $1/(2n+2) < A_n < 1/(2n-2)$

Sol. (i) $0 < \tan x < 1$, when $0 < x < \pi/4$, we have
$0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$



(ii) we have $A_n = \int_0^{\pi/4}(\tan x)^n\ dx$

$\Rightarrow \int_0^{\pi/4}(\tan x)^{n+1}\ dx < \int_0^{\pi/4}(\tan x)^n\ dx \Rightarrow A_{n+1} < A_n$

Now, for $n > 0$, $A_n + A_{n+2} = \int_0^{\pi/4}(\tan x)^n + (\tan x)^{n+2}]\ dx$

$= \int_0^{\pi/4} (\tan x)^n (\sec^2 x) dx = \left[\frac{1}{(n+1)}(\tan x)^{n+1}\right]_0^{\pi/4} = \frac{1}{(n+1)}(1-0)$.

Similarly $A_n + A_{n-2} = \frac{1}{n-1}$

since $A_{n+2} < A_{n+1} < A_n$, we get $A_n + A_{n+2} < 2A_n$

$\Rightarrow \quad \frac{1}{n+1} < 2A_n \quad \Rightarrow \quad \frac{1}{2n+2} < A_n$ .... (1)

Also for $n > 2$, $A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$

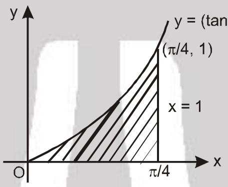
$\Rightarrow \quad 2A_n < \frac{1}{n-1}$ .... (2)

$\Rightarrow \quad A_n < \frac{1}{2n-2}$

Combining (1) and (2) we get Hence Proved.

Hindi. (i) $\because$ $0 < \tan x < 1$, जब $0 < x < \pi/4$

सभी $n \in N$ के लिए $0 < (\tan x)^{n+1} < (\tan x)^n$



(ii) दिया है $A_n = \int_0^{\pi/4} (\tan x)^n \, dx$

$\Rightarrow \int_0^{\pi/4} (\tan x)^{n+1} \, dx < \int_0^{\pi/4} (\tan x)^n \, dx \Rightarrow A_{n+1} < A_n$

$n > 0$ के लिए,

$A_n + A_{n+2} = \int_0^{\pi/4} (\tan x)^n + (\tan x)^{n+2}] \, dx = \int_0^{\pi/4} (\tan x)^n \, (\sec^2 x) \, dx \left[\frac{1}{(n+1)}(\tan x)^{n+1}\right]_0^{\pi/4}$

$= \frac{1}{(n+1)}(1-0)$.

समानतः $A_n + A_{n-2} = \frac{1}{n-1}$

since $A_{n+2} < A_{n+1} < A_n$, we get $A_n + A_{n+2} < 2A_n$

$\Rightarrow \quad \frac{1}{n+1} < 2A_n \quad \Rightarrow \quad \frac{1}{2n+2} < A_n$ .... (1)

$n > 2$ के लिए, $A_n + A_n < A_n + A_{n-2} = \frac{1}{n-1}$

$\Rightarrow \quad 2A_n < \frac{1}{n-1}$ .... (2)

$\Rightarrow \quad A_n < \frac{1}{2n-2}$

(1) तथा (2) को एक साथ लेने पर

अतः सिद्ध हुआ।

PART - II : ONLY ONE OPTION CORRECT TYPE

भाग - II : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

SECTION (A) : D.I. IN TERMS OF INDEFINITE INTIGRATION, USING SUBSTITUTION AND BY PARTS
खण्ड (A) : प्रतिस्थापन तथा खण्डशः समाकलन की सहायता से अनिश्चित समाकलन के रूप में निश्चित समाकलन

A-1. If $\int_1^x \frac{dt}{|t|\sqrt{t^2-1}} = \frac{\pi}{6}$, then x can be equal to :

(A*) $\frac{2}{\sqrt{3}}$ (B) $\sqrt{3}$ (C) 2 (D) $\frac{4}{\sqrt{3}}$

यदि $\int_1^x \frac{dt}{|t|\sqrt{t^2-1}} = \frac{\pi}{6}$ हो, तब x बराबर हो सकता है –

(A) $\frac{2}{\sqrt{3}}$ के (B) $\sqrt{3}$ के (C) 2 के (D) $\frac{4}{\sqrt{3}}$

Sol. $\int_1^x \frac{dx}{|t|\sqrt{t^2-1}} = \frac{\pi}{6} \Rightarrow \left[\sec^{-1}t\right]_1^x = \frac{\pi}{6} = \sec^{-1}x - \sec^{-1}1 = \frac{\pi}{6} = \sec^{-1}x = \frac{\pi}{6} \Rightarrow x = \sec^{-1}\frac{\pi}{6} = \frac{2}{\sqrt{3}}$

A-2. The value of the integral $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$, where $0 < \alpha < \frac{\pi}{2}$, is equal to: **[16JM120529]**

समाकलन $\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1}$, जहाँ $0 < \alpha < \frac{\pi}{2}$, का मान है–

(A) $\sin\alpha$ (B) $\alpha\sin\alpha$ (C*) $\frac{\alpha}{2\sin\alpha}$ (D) $\frac{\alpha}{2}\sin\alpha$

Sol. $I = \int_0^1 \frac{dx}{x^2+2x\cos\alpha+\sin^2\alpha+\cos^2\alpha} = \int_0^1 \frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha} = \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{x+\cos\alpha}{\sin\alpha}\right)\right]_0^1 = \frac{\alpha}{2\sin\alpha}$

A-3. If $f(x) = \begin{cases} x & x<1 \\ x-1 & x\geq 1 \end{cases}$, then $\int_0^2 x^2 f(x)\ dx$ is equal to :

यदि $f(x) = \begin{cases} x & x<1 \\ x-1 & x\geq 1 \end{cases}$, तब $\int_0^2 x^2 f(x)\ dx$ का मान है–

(A) 1 (B) $\frac{4}{3}$ (C*) $\frac{5}{3}$ (D) $\frac{5}{2}$

Sol. $\int_0^2 x^2 f(x)\ dx = \int_0^1 x^3\ dx + \int_1^2 (x^3-x^2)\ dx = \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^4}{4}-\frac{x^3}{3}\right]_1^2$

$= \frac{1}{4}+\left[4-\frac{8}{3}-\left(\frac{1}{4}-\frac{1}{3}\right)\right] = 4 - \frac{8}{3}+\frac{1}{3} = 4-\frac{1}{3} = \frac{5}{3}$

A-4. If f(0) = 1, f(2) = 3, f′(2) = 5 and f′(0) is finite, then $\int_0^1 x\ .\ f''(2x)\ dx$ is equal to **[16JM120530]**

(A) zero (B) 1 (C*) 2 (D) 3

यदि f(0) = 1, f(2) = 3, f′(2) = 5 तथा f′(0) परिमित है, तब $\int_0^1 x\ .\ f''(2x)\ dx$ का मान है–

(A) शून्य (B) 1 (C) 2 (D) 3

Sol. $I = \int_0^1 \underset{I}{x}\,\underset{II}{f''(2x)}\ dx = \left[\frac{x\ f'(2x)}{2} - \frac{f(2x)}{4}\right]_0^1 \quad = \frac{f'(2)}{2} - \frac{f(2)}{4} + \frac{f(0)}{4} = \frac{5}{2} - \frac{3}{4} + \frac{1}{4} = 2$

A-5. $\int_0^\pi |1 + 2\cos x|\ dx$ is equal to :

$\int_0^\pi |1 + 2\cos x|\ dx$ का मान है–

(A) $\frac{2\pi}{3}$ (B) π (C) 2 (D*) $\frac{\pi}{3} + 2\sqrt{3}$

Sol. $\int_0^\pi |1+2\cos x|\ dx = \int_0^{2\pi/3} (1+2\cos x)dx - \int_{2\pi/3}^{\pi} (1+2\cos x)\,dx$

$= \frac{2\pi}{3} + 2\ .\ \left(\frac{\sqrt{3}}{2}\right) - \left[\pi - \left(\frac{2\pi}{3} + 2\ .\ \frac{\sqrt{3}}{2}\right)\right] = \frac{2\pi}{3} - \frac{\pi}{3} + \sqrt{3} + \sqrt{3} = \frac{\pi}{3} + 2\sqrt{3}$

A-6. The value of $\int_{-1}^{3} (|x-2| + [x])dx$ is ([x] stands for greatest integer less than or equal to x) **[16JM120531]**

$\int_{-1}^{3} (|x-2| + [x])dx$ का मान (जहाँ [x], x से छोटा या बराबर महत्तम पूर्णांक को प्रदर्शित करता है) है–

(A*) 7 (B) 5 (C) 4 (D) 3

Sol. $\int_{-1}^{3} |x-2| + [x]\ dx = \int_{-1}^{0} (2-x-1)\ dx + \int_0^1 (2-x)\ dx + \int_1^2 (2-x)+1\ dx + \int_2^3 (x-2+2)\ dx$

$= \left[x - \frac{x^2}{2}\right]_{-1}^{0} + \left[2x - \frac{x^2}{2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2}\right]_2^3 = -\left(-1-\frac{1}{2}\right) + \left(2 - \frac{1}{2}\right) + \left((6-2) - \left(3 - \frac{1}{2}\right)\right) + \frac{9}{2} - 2$

$= \frac{3}{2} + \frac{3}{2} + 4 - \frac{5}{2} + \frac{5}{2} = 7$

A-7. $\int_0^\infty [2e^{-x}]\ dx$, where [.] denotes the greatest integer function, is equal to :

$\int_0^\infty [2e^{-x}]\ dx$, जहाँ [·] महत्तम पूर्णांक फलन को दर्शाता है, का मान है–

(A) 0 (B*) $\ell n\ 2$ (C) e^2 (D) $2e^{-1}$

Sol. $I = \int_0^\infty \left[2\ e^{-x}\right]\ dx \quad \therefore \quad 2e^{-x}$ decreases for $x \in [0, \infty) \quad \Rightarrow \quad 0 < 2e^{-x} \le 2\ \forall\ x \in [0, \infty)$

for $x > \ell n2,\ [2e^{-x}] = 0 \Rightarrow I = \int_0^{\ell n2} [2e^{-x}]\ dx + \int_{\ell n\ 2}^{\infty} [2e^{-x}]\ dx = \int_0^{\ell n2} 1\ .\ dx + \int_{\ell n2}^{\infty} 0.dx = \ell n2$

Hindi $I = \int_0^\infty \left[2\ e^{-x}\right]\ dx \quad \therefore \quad x \in [0, \infty)$ के लिए $2e^{-x}$ ह्रासमान है

$\Rightarrow \quad 0 < 2e^{-x} \le 2\ \forall\ x \in [0, \infty)$

$x > \ell n2$ के लिए $[2e^{-x}] = 0$

$\Rightarrow I = \int_0^{\ell n2} [2e^{-x}]\ dx + \int_{\ell n\ 2}^{\infty} [2e^{-x}]\ dx = \int_0^{\ell n2} 1\ .\ dx + \int_{\ell n2}^{\infty} 0.dx = \ell n2$

A-8. $\displaystyle\int_{\ell n\pi-\ell n2}^{\ell n\pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}$ dx is equal to

$\displaystyle\int_{\ell n\pi-\ell n2}^{\ell n\pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}$ dx का मान है–

(A*) $\sqrt{3}$ (B) $-\sqrt{3}$ (C) $\frac{1}{\sqrt{3}}$ (D) $-\frac{1}{\sqrt{3}}$

Sol. $$\int_{\ell n\pi-\ell n2}^{\ell n\pi} \frac{e^x}{1-\cos\left(\frac{2}{3}e^x\right)}dx = \frac{3}{2}\int_{\pi/3}^{2\pi/3}\frac{dt}{1-\cos t} = \frac{3}{2}\int_{\pi/3}^{2\pi/3}\frac{dt}{1-\left(1-2\sin^2\frac{t}{2}\right)}$$

$$= \frac{3}{2.2}\int_{\pi/3}^{2\pi/3}\operatorname{cosec}^2\frac{t}{2}dt = \frac{3}{2}\left[-\cot\frac{t}{2}\right]_{\pi/3}^{2\pi/3} = -\frac{3}{2}\left[\sqrt{3}-\frac{1}{\sqrt{3}}\right] = \frac{3}{4}\left(\frac{4}{\sqrt{3}}\right) = \sqrt{3}$$

A-9. If $I_1 = \displaystyle\int_e^{e^2}\frac{dx}{\ell n\ x}$ and $I_2 = \displaystyle\int_1^2\frac{e^x}{x}$ dx, then **[16JM120532]**

(A*) $I_1 = I_2$ (B) $2\ I_1 = I_2$ (C) $I_1 = 2\ I_2$ (D) $I_1 + I_2 = 0$

यदि $I_1 = \displaystyle\int_e^{e^2}\frac{dx}{\ell n\ x}$ और $I_2 = \displaystyle\int_e^{e^2}\frac{dx}{\ell n\ x}$ dx है, तब–

(A*) $I_1 = I_2$ (B) $2\ I_1 = I_2$ (C) $I_1 = 2\ I_2$ (D) $I_1 + I_2 = 0$

Sol. $I_1 = \displaystyle\int_e^{e^2}\frac{dx}{\ell nx}$

Let $\ell nx = t$ ⇒ $x = e^t$ ⇒ $dx = e^t dt$ $\quad I_1 = \displaystyle\int_1^2\frac{e^t}{t}dt = I_2$

Hindi $I_1 = \displaystyle\int_e^{e^2}\frac{dx}{\ell nx}$

माना $\ell nx = t$ ⇒ $x = e^t$ ⇒ $dx = e^t dt$ $\quad I_1 = \displaystyle\int_1^2\frac{e^t}{t}dt = I_2$

A-10. $\displaystyle\int_0^{\pi/4}\frac{x\,.\sin x}{\cos^3 x}$ dx equals to :

(A) $\frac{\pi}{4}+\frac{1}{2}$ (B*) $\frac{\pi}{4}-\frac{1}{2}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{4}+1$

$\displaystyle\int_0^{\pi/4}\frac{x\,.\sin x}{\cos^3 x}$ dx का मान है–

(A) $\frac{\pi}{4}+\frac{1}{2}$ (B*) $\frac{\pi}{4}-\frac{1}{2}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{4}+1$

Sol. $$\int_0^{\pi/4}\frac{x\,.\sin x}{\cos^3 x}dx = \int_0^{\pi/4}x.\tan x.\sec^2 x\ dx$$

Applying by parts खण्डशः समाकल से

$$x\left(\frac{\tan^2 x}{2}\right)_0^{\pi/4} - \int_0^{\pi/4}\left(\frac{\tan^2 x}{2}\right)dx$$

After solving we get हल करने पर $= \frac{\pi}{4} - \frac{1}{2}$

A-11. The value of the definite integral $\int_{\frac{3}{2}}^{\frac{9}{4}}\left[\sqrt{2x-\sqrt{5(4x-5)}} + \sqrt{2x+\sqrt{5(4x-5)}}\right]dx$ is equal to

निश्चित समाकल $\int_{\frac{3}{2}}^{\frac{9}{4}}\left[\sqrt{2x-\sqrt{5(4x-5)}} + \sqrt{2x+\sqrt{5(4x-5)}}\right]dx$ का मान है – **[16JM120533]**

(A) $4\sqrt{5} - \frac{2\sqrt{2}}{5}$ (B) $4\sqrt{5}$ (C) $4\sqrt{3} - \frac{4}{3}$ (D*) $\frac{3\sqrt{5}}{\sqrt{8}}$

Sol. Let माना $5(4x – 5) = t^2$

$\Rightarrow \quad 20x – 25 = t^2 \quad \Rightarrow \quad x = \frac{t^2+25}{20}$

Also $\quad 20\,dx = 2t\,dt \quad$ or या $\quad dx = \frac{t}{10}\,dt$

$$\therefore \quad I = \int_{\sqrt{5}}^{\sqrt{20}}\left(\sqrt{\frac{t^2+25}{10}-t} + \sqrt{\frac{t^2+25}{10}+t}\right)\frac{t}{10}\,dt = \int_{\sqrt{5}}^{\sqrt{20}}\left(\frac{|t-5|}{\sqrt{10}} + \frac{|t+5|}{\sqrt{10}}\right)\frac{t}{10}\,dt$$

$$= \int_{\sqrt{5}}^{\sqrt{20}}\left(\frac{-t+5}{\sqrt{10}} + \frac{t+5}{\sqrt{10}}\right)\frac{t}{10}\,dt = \frac{1}{\sqrt{10}}\left[\frac{t^2}{2}\right]_{\sqrt{5}}^{\sqrt{20}} = \frac{1}{\sqrt{10}}\left[\frac{20}{2} - \frac{5}{2}\right] = \frac{3\sqrt{5}}{\sqrt{8}}$$

A-12. If $\int_{\ell n2}^{x}\frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$, then x is equal to

(A) 4 (B) ℓn 8 (C*) ℓn 4 (D) ℓn 2

यदि $\int_{\ell n2}^{x}\frac{dx}{\sqrt{e^x-1}} = \frac{\pi}{6}$, तब x का मान है–

(A) 4 (B) ℓn 8 (C*) ℓn 4 (D) ℓn 2

Sol. $\because \quad I = \int\frac{dx}{\sqrt{e^x-1}} = \int\frac{e^{x/2}\,dx}{e^{x/2}\sqrt{(e^{x/2})^2-1}}$

Let $\quad e^{x/2} = z \quad \Rightarrow \quad I = 2\int\frac{dz}{|z|\sqrt{z^2-1}} = 2\sec^{-1}z + c = 2\sec^{-1}(e^{x/2}) + c$

Given $\quad \frac{\pi}{6} = \int_{\ell n2}^{x}\frac{dx}{\sqrt{e^x-1}} = 2\left[\sec^{-1}(e^{x/2}) - \sec^{-1}\sqrt{2}\,\right]$

$\therefore \quad \sec^{-1}(e^{x/2}) = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3} \quad \therefore \quad e^{x/2} = 2 \quad \Rightarrow \quad \frac{x}{2} = \ell n\,2 \quad \Rightarrow \quad x = \ell n\,4$

Hindi $\because \quad I = \int\frac{dx}{\sqrt{e^x-1}} = \int\frac{e^{x/2}\,dx}{e^{x/2}\sqrt{(e^{x/2})^2-1}}$

माना $e^{x/2} = z$ ⇒ $I = 2\int \frac{dz}{|z|\sqrt{z^2-1}} = 2\sec^{-1}z + c = 2\sec^{-1}(e^{x/2}) + c$

दिया गया है $\frac{\pi}{6} = \int_{\ell n2}^{x} \frac{dx}{\sqrt{e^x-1}} = 2\,[\sec^{-1}(e^{x/2}) - \sec^{-1}\sqrt{2}\ ]$

∴ $\sec^{-1}(e^{x/2}) = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$ ∴ $e^{x/2} = 2$ ⇒ $\frac{x}{2} = \ell n\, 2$ ⇒ $x = \ell n\, 4$

A-13. $\int_0^{\infty} \frac{x^2+1}{x^4+7x^2+1}\, dx =$ **[16JM120534]**

(A) π (B) $\frac{\pi}{2}$ (C*) $\frac{\pi}{3}$ (D) $\frac{\pi}{6}$

Sol. Dividing Nʳ & Dʳ by x² we get अंश व हर में x² से विभाजित करने पर

$$\int_0^{\infty} \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+7}dx = \int_0^{\infty} \frac{\left(1+\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2+9},$$ substitute $x - \frac{1}{x} = t$ रखने पर

$$= \int_{-\infty}^{\infty} \frac{dt}{t^2+9} = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\Bigg|_{-\infty}^{\infty} = \frac{\pi}{3}$$

Section (B) : Definite Integration using Properties
खण्ड (B) : गुणधर्मो की सहायता से निश्चित समाकलन

B-1. Suppose for every integer n, . $\int_n^{n+1} f(x)dx = n^2$ The value of $\int_{-2}^{4} f(x)dx$ is :

माना प्रत्येक पूर्णाक n के लिए $\int_n^{n+1} f(x)dx = n^2$ हो, तो $\int_{-2}^{4} f(x)dx$ का मान है–

(A) 16 (B) 14 (C*) 19 (D) 21

Sol. $I = \int_{-2}^{-1} f(x)\ dx + \int_{-1}^{0} f(x)\ dx + \int_{0}^{1} f(x)\ dx + \int_{1}^{2} f(x)\ dx + \int_{2}^{3} f(x)\ dx + \int_{3}^{4} f(x)\ dx$

$= (-2)^2 + (-1)^2 + 0 + 1^2 + 2^2 + 3^2$

[using given relation] [दिये गए संबंध के प्रयोग से]

$= 19$

B-2. Let $f : R \to R$, $g : R \to R$ be continuous functions. Then the value of integeral **[16JM120535]**

$$\int_{\ell n\lambda}^{\ell n 1/\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x)+g(-x)]}\ dx \text{ is :}$$

(A) depend on λ (B) a non-zero constant (C*) zero (D) 2

माना $f : R \to R$, $g : R \to R$ सतत् फलन है। तब समाकल $\int_{\ell n\lambda}^{\ell n 1/\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)[g(x)+g(-x)]}$ dx का मान है–

(A) λ पर निर्भर (B) एक अशून्य अचर (C) शून्य (D) 2

Sol. $I = \displaystyle\int_{\ell n\lambda}^{-\ell n\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)\left[g(x)+g(-x)\right]}dx = 0$

Since चूँकि $\displaystyle\int_{\ell n\lambda}^{-\ell n\lambda} \frac{f\left(\frac{x^2}{4}\right)[f(x)-f(-x)]}{g\left(\frac{x^2}{4}\right)\left[g(x)+g(-x)\right]}dx = 0$ is an odd function (एक विषम फलन है)

B-3. $\displaystyle\int_{-1}^{1} \cot^{-1}\left(\frac{x+x^3}{1+x^4}\right)dx$ is equal to बराबर है

(A) 2π (B) $\frac{\pi}{2}$ (C) 0 (D*) π

Sol. $I = \displaystyle\int_{0}^{1}\left\{\cot^{-1}\left(\frac{x+x^3}{1+x^4}\right)+\cot^{-1}\left(\frac{-x-x^3}{1+x^4}\right)\right\}dx = \int_0^1 \pi dx = \pi$

B-4. $\displaystyle\int_{-2}^{0} \{x^3+3x^2+3x+3+(x+1)\cos(x+1)\}\ dx$ is equal to **[16JM120536]**

$\displaystyle\int_{-2}^{0} \{x^3+3x^2+3x+3+(x+1)\cos(x+1)\}\ dx$ का मान है–

(A) – 4 (B) 0 (C*) 4 (D) 6

Sol. Put x + 1 = t

$\displaystyle\int_{-1}^{1} (t^3 + 2 + t\cos t)\ dt$ (∵ t^3 and t cos t are odd functions)

$= 2\displaystyle\int_0^1 2\ dt = 4$

Hindi x + 1 = t रखने पर

$\displaystyle\int_{-1}^{1} (t^3 + 2 + t\cos t)\ dt$ (∵ t^3 एवं t cos t विषम फलन है)

$= 2\displaystyle\int_0^1 2\ dt = 4$

B-5. $\displaystyle\int_{-1}^{1} x\ell n(1+e^x)dx =$

(A) 0 (B) ℓn(1 + e) (C) ℓn(1 + e) – 1 (D*) 1/3

Sol. $I = \displaystyle\int_0^1 [x\ell n(1+e^x)+\{-x\ell n(1+e^{-x})\}]dx = \int_0^1 [x\ell n(1+e^x)-x\ell n(1+e^x)+x^2]dx = \int_0^1 x^2\,dx = \frac{1}{3}$

B-6. If $\displaystyle\int_{-1}^{3/2} |\,x\sin\pi x\,|\ dx = \frac{k}{\pi^2}$, then the value of k is : **[16JM120537]**

यदि $\displaystyle\int_{-1}^{3/2} |\,x\sin\pi x\,|\ dx = \frac{k}{\pi^2}$ हो, तो k का मान है–

(A*) $3\pi + 1$ (B) $2\pi + 1$ (C) 1 (D) 4

Sol. $I = \int_{-1}^{0} x\sin\pi x \ dx + \int_{0}^{1} x\sin\pi x \ dx - \int_{1}^{3/2} x\sin\pi x \ dx = 2\int_{0}^{1} x\sin\pi x \ dx - \int_{1}^{3/2} x\sin\pi x \ dx$

$= \left[\frac{-2\times\cos\pi x}{\pi} + \frac{2\sin\pi x}{\pi^2}\right]_0^1 - \left[\frac{-x\cos\pi x}{\pi} + \frac{\sin\pi x}{\pi^2}\right]_1^{3/2} = \frac{3\pi+1}{\pi^2} = \frac{k}{\pi^2} \Rightarrow k = 3\pi + 1$

B-7. The value of definite integral is $\int_0^{\frac{\pi^2}{4}} \frac{dx}{1+\sin\sqrt{x}+\cos\sqrt{x}}$

समाकल $\int_0^{\frac{\pi^2}{4}} \frac{dx}{1+\sin\sqrt{x}+\cos\sqrt{x}}$ का मान है

(A) π ln 2 (B*) $\frac{\pi \ln 2}{2}$ (C) $\frac{\pi \ln 2}{4}$ (D) 2π ln2

Sol. Let माना $x = t^2$

$\therefore \quad I = \int_0^{\pi/2} \frac{2tdt}{1+\sin t+\cos t}$; $I = \int_0^{\pi/2} \frac{\pi - 2t}{1+\cos t+\sin t}dt$

$2I = \pi \int_0^{\pi/2} \frac{dt}{(1+\cos t)+\sin t}$

$I = \frac{\pi}{4}\int_0^{\pi/2} \frac{1}{\cos^2\frac{t}{2}+\cos\frac{t}{2}\sin\frac{t}{2}}dt = \frac{\pi}{4}\int_0^{\pi/2} \frac{\sec^2\frac{t}{2}}{1+\tan\frac{t}{2}}dt = \frac{\pi}{2}\ln\left(1+\tan\frac{t}{2}\right)\Big|_0^{\pi/2} = \frac{\pi\ell n2}{2}$

B-8. $\int_{2-\ell n 3}^{3+\ell n 3} \frac{\ell n\ (4+x)}{\ell n\ \ (4+x)+\ell n\ \ (9-x)}$ dx is equal to : **[16JM120538]**

(A) cannot be evaluated (B) is equal to $\frac{5}{2}$

(C) is equal to $1 + 2\ \ell n\ 3$ (D*) is equal to $\frac{1}{2} + \ell n\ 3$

$\int_{2-\ell n 3}^{3+\ell n 3} \frac{\ell n\ (4+x)}{\ell n\ \ (4+x)+\ell n\ \ (9-x)}$ dx का मान है–

(A) गणना नहीं की जा सकती (B) $\frac{5}{2}$ के बराबर है

(C) $1 + 2\ \ell n\ 3$ के बराबर है (D) $\frac{1}{2} + \ell n\ 3$ के बराबर है

Sol. $I = \int_{2-\ell n3}^{3+\ell n3} \frac{\ell n(4+x)}{\ell n(4+x)+\ell n(9-x)}dx = I = \int_{2-\ell n3}^{3+\ell n3} \frac{\ell n(9-x)}{\ell n(9-x)+\ell n(4+x)}dx$

$2I = \int_{2\ -\ \ell n3}^{3\ +\ \ell n3} 1.\ \ dx = 3 + \ell n\ 3 - (2 - \ell n\ 3) = 1 + 2\ \ell n3$

B-9. The value of the definite integral $I = \int_0^{\pi} x\sqrt{1+|\cos x|}\ \ dx$ is equal to

निश्चित समाकल $I = \int_0^{\pi} x\sqrt{1+|\cos x|}\ \ dx$ का मान है

(A) $2\sqrt{2}\ \ \pi$ (B) $\sqrt{2}\ \ \pi$ (C*) 2 π (D) 4π

Sol. $I = \int_0^{\pi} (\pi - x)\sqrt{1+|\cos x|}\,dx \Rightarrow 2I = \pi\int_0^{\pi}\sqrt{1+|\cos x|}\,dx \Rightarrow 2I = 2\pi\int_0^{\pi/2}\sqrt{1+\cos x}\ dx$

$\Rightarrow I = \sqrt{2}\,\pi = \int_0^{\pi/2}\cos\left(\frac{x}{2}\right)dx = 2\sqrt{2}\,\pi.\sin\left(\frac{x}{2}\right)\Big|_0^{\pi/2} = 2\pi$

B-10. The value of $\int_0^{\pi/2} \ell n\,|\tan x + \cot x|\ dx$ is equal to :

$\int_0^{\pi/2} \ell n\,|\tan x + \cot x|\ \ dx$ का मान है–

(A*) $\pi\ \ell n\ 2$ (B) $-\pi\ \ell n\ 2$ (C) $\frac{\pi}{2}\ \ell n\ 2$ (D) $-\ \frac{\pi}{2}\ell n\ 2$

Sol. $\int_0^{\pi/2} \ell n\ \ |\tan x + \cot x|\,dx = \int_0^{\pi/2} \ell n\ \left|\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right|\ dx = \int_0^{\pi/2} -\ell n\ \left|\left(\frac{\sin 2x}{2}\right)\right|\ dx$

$= -\int_0^{\pi/2} \ell n\left|\left(\frac{\sin 2x}{2}\right)\right| dx = -\int_0^{\pi/2} \ell n \sin 2x\ \ dx + \int_0^{\pi/2} \ell n 2\,dx$

$= \left(\frac{\pi}{2}\ell n\ \ 2\right) + \ell n\,2\ .\ \left(\frac{\pi}{2}\right) = \pi\ \ell n\ 2$

B-11. Let $I_1 = \int_0^1 \frac{e^x dx}{1+x}$ and $I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$, then $\frac{I_1}{I_2}$ is

माना $I_1 = \int_0^1 \frac{e^x dx}{1+x}$ तथा $I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$, तब $\frac{I_1}{I_2}$ का मान है –

(A) 3/e (B) e/3 (C*) 3e (D) 1/3e

Sol. $I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$

Let $1 - x^3 = t \Rightarrow -\ 3x^2 dx = dt$

$\Rightarrow I_2 = \frac{1}{3}\int_0^1 \frac{dt}{e^{1-t}(1+t)} = \frac{1}{3e}\int_0^1 \frac{e^t dt}{1+t} = \frac{I_1}{3e} \Rightarrow \frac{I_1}{I_2} = 3e.$

Hindi $I_2 = \int_0^1 \frac{x^2 dx}{e^{x^3}(2-x^3)}$

माना $1 - x^3 = t \Rightarrow -\ 3x^2 dx = dt$

$\Rightarrow I_2 = \frac{1}{3}\int_0^1 \frac{dt}{e^{1-t}(1+t)} = \frac{1}{3e}\int_0^1 \frac{e^t dt}{1+t} = \frac{I_1}{3e} \Rightarrow \frac{I_1}{I_2} = 3e.$

B-12. The value of $\int_0^{[x]} \{x\}\ \ dx$ (where [.] and { . } denotes greatest integer and fraction part function respectively) is **[16JM120539]**

(A*) $\frac{1}{2}[x]$ (B) 2[x] (C) $\frac{1}{2[x]}$ (D) [x]

$\int_0^{[x]} \{x\}\ \ dx$ का मान (जहाँ [.] तथा { . } क्रमशः महत्तम पुर्णांक व भिन्नात्मक भाग फलन को प्रदर्शित करते हैं) हैं –

(A*) $\frac{1}{2}[x]$ (B) 2[x] (C) $\frac{1}{2[x]}$ (D) [x]

Sol. $\int_0^{na} f(x)\ dx = n\int_0^{a} f(x)\ \ dx$

where a is period of f(x).

$\therefore \qquad I = \int_0^{[x]} \{x\}\ dx \quad = [x]\int_0^{1} x\ dx \qquad \{\because \quad \{x\} = x, \text{ when } x \in [0, 1)\} \qquad \Rightarrow \qquad I = \frac{[x]}{2}$

Hindi $\int_0^{na} f(x)\ dx = n\int_0^{a} f(x)\ \ dx$

जहाँ a , f(x) का आवर्तकाल है।

$\therefore \qquad I = \int_0^{[x]} \{x\}\ dx \quad = [x]\int_0^{1} x\ dx \qquad \{\because \quad \{x\} = x,$ जब $x \in [0, 1)\} \qquad \Rightarrow \qquad I = \frac{[x]}{2}$

B-13. If $\int_0^{11} \frac{11^x}{11^{[x]}}\ dx = \frac{k}{\log 11}$, (where [ ] denotes greatest integer function) then value of k is **[16JM120540]**

(A) 11 (B) 101 (C*) 110 (D)121

यदि $\int_0^{11} \frac{11^x}{11^{[x]}}\ dx = \frac{k}{\log 11}$ (जहाँ [ ] महत्तम पूर्णांक फलन को दर्शाता है), तब k का मान है–

(A) 11 (B) 101 (C*) 110 (D)121

Sol. $I = \int_0^{11} 11^{\{x\}} dx \ = \int_0^{11\times 1} 11^{\{x\}} dx = 11\int_0^{1} 11^{\{x\}} dx \qquad \{\because \ \{x\} \text{ is periodic with period } 1\ \}$

$\{\because \ \{x\}$ आवर्ती है जिसका आवर्तकाल 1 है$\}$

$= 11\int_0^{1} 11^x dx = 11\left[\frac{11^x}{\ell n 11}\right]_0^1 \ = \ 11\left[\frac{11}{\ell n 11} - \frac{1}{\ell n 11}\right] = \frac{110}{\ell n 11} = \frac{k}{\ell n 11} \qquad k = 110$

Section (C) : Leibnitz formula and Wallis' formula

खण्ड (C) : लेबनीज सूत्र एवं वॉलीस सूत्र (Wallis' Formula)

C-1. $f(x) = \int_x^{x^2} \frac{e^t}{t} dt$, then तब f ' (1) is equal to बराबर है :

(A*) e (B) 2e (C) $2e^2 - 2$ (D) $e^2 - e$

Sol. $f'(x) = \frac{e^{x^2}}{x^2}.2x - \frac{e^x}{x} = \frac{2e^{x^2} - e^x}{x} \ \Rightarrow \qquad f'(1) = 2e - e = e$

C-2. $f(x) = \int_0^{x} (t-1)(t-2)^2(t-3)^3(t-4)^5\ dt$ (x > 0) then number of points of extremum of f(x) is**[16JM120541]**

$f(x) = \int_0^{x} (t-1)(t-2)^2(t-3)^3(t-4)^5\ dt$ (x > 0) तब f(x) के चरम होने के लिए बिन्दुओं की संख्या है।

(A) 4 (B*) 3 (C) 2 (D) 1

Sol. $f'(x) = (x-1)(x-2)^2(x-3)^3(x-4)^5$

$$\underset{1}{\overset{-}{}}\overset{+}{}\underset{2}{\overset{+}{}}\underset{3}{\overset{-}{}}\underset{4}{\overset{+}{}}$$

Hence x = 1, 3, 4 are points of extermum. अतः x = 1, 3, 4 चरम बिन्दु है।

C-3. $\underset{h\to 0}{\text{Limit}} \dfrac{\int_a^{x+h} \ell n^2 t\ dt - \int_a^{x} \ell n^2 t\ dt}{h}$ equals to :

(A) 0 (B*) $\ell n^2 x$ (C) $\dfrac{2\ell n x}{x}$ (D) does not exist

$\underset{h\to 0}{\text{Limit}} \dfrac{\int_a^{x+h} \ell n^2 t\ dt - \int_a^{x} \ell n^2 t\ dt}{h}$ का मान है–

(A) 0 (B*) $\ell n^2 x$ (C) $\dfrac{2\ell n x}{x}$ (D) विद्यमान नहीं है।

Sol. $I = \underset{h\to 0}{\text{Limit}} \dfrac{\int_a^{x} \ell n^2 t\ dt + \int_x^{x+h} \ell n^2 t\ dx - \int_a^{x} \ell n^2 t\ dt}{h}$ $I = \underset{h\to 0}{\text{Limit}} \dfrac{\int_x^{x+h} \ell n^2 t\ dt}{h}$

Using L hospital we get

$I = \underset{h\to 0}{\text{Limit}}\ \ell n^2(x+h) = \ell n^2 x$

Hindi. $I = \underset{h\to 0}{\text{Limit}} \dfrac{\int_a^{x} \ell n^2 t\ dt + \int_x^{x+h} \ell n^2 t\ dx - \int_a^{x} \ell n^2 t\ dt}{h}$ $I = \underset{h\to 0}{\text{Limit}} \dfrac{\int_x^{x+h} \ell n^2 t\ dt}{h}$

L - Hospital के प्रयोग से

$I = \underset{h\to 0}{\text{Limit}}\ \ell n^2(x+h) = \ell n^2 x$

C-4. The value of the function $f(x) = 1 + x + \int_1^x (\ell n^2 t + 2\,\ell n t)dt$, where $f'(x)$ vanishes is: **[16JM120542]**

जहाँ $f'(x)$ शून्य होता है वहाँ फलन $f(x) = 1 + x + \int_1^x (\ell n^2 t + 2\,\ell n t)dt$ का मान है–

(A) e^{-1} (B) 0 (C) $2e^{-1}$ (D*) $1 + 2e^{-1}$

Sol. $f'(x) = 0$

$\Rightarrow$ $1 + \ell n^2 x + 2\ell n\, x = 0$ $\Rightarrow$ put $\ell nx = p$ रखने पर $\Rightarrow p^2 + 2p + 1 = 0$

$\Rightarrow$ $p = -1$ $\Rightarrow$ $\ell n\, x = -1$ $\Rightarrow$ $x = \dfrac{1}{e}$ $\Rightarrow$ $f(x) = \dfrac{1}{e} + \int_1^{1/e} \ell n^2\ t\ \ dt + 2\int_1^{1/e} \ell n\ \ t\ \ dt$

Let (माना) $\ell n\, t = z$ $\Rightarrow$ $\dfrac{1}{t}dt = dz$ $\Rightarrow$ $dt = e^z\, dz$

$$\therefore\ f\left(\frac{1}{e}\right) = 1 + \frac{1}{e} + \int_0^{-1} z^2\ \ e^z\ \ dz + 2\int_0^{-1} ze^z dz = 1 + \frac{1}{e} - \left[z^2e^z - \int 2\ \ ze^z dz\right]_{-1}^{0} - 2\int_{-1}^{0} ze^z dz$$

$$= 1 + \frac{1}{e} - \left[z^2e^z\right]_{-1}^{0} = 1 + \frac{1}{e} + \frac{1}{e} = 1 + \frac{2}{e}$$

C-5. If $\int_a^y \cos t^2 dt = \int_a^{x^2} \frac{\sin t}{t} dt$, then the value of $\frac{dy}{dx}$ is

यदि $\int_a^y \cos t^2 dt = \int_a^{x^2} \frac{\sin t}{t} dt$, हो तो $\frac{dy}{dx}$ का मान है–

(A) $\frac{2\sin^2 x}{x\cos^2 y}$ (B*) $\frac{2\sin x^2}{x\cos y^2}$ (C) $\frac{2\sin x^2}{x\left(1-2\sin\frac{y^2}{2}\right)}$ (D) $\frac{\sin x^2}{2y}$

Sol. $\int_a^y \cos\ t^2\ dt = \int_a^{x^2} \frac{\sin t}{t} dt$

differentiating both sides w.r.t x we get दोनों तरफ x के सापेक्ष अवकलन करने पर

$$\frac{d}{dx}\int_a^y \cot\ t^2\ dt = \frac{d}{dx}\int_a^{x^2}\frac{\sin t}{t}dt$$

$$\text{RHS} = \frac{\sin\ [x^2]}{x^2}\ \frac{dx^2}{dx} = 2\times\frac{\sin x^2}{x^2}$$

$$\text{L.H.S.} = \frac{d}{dy}\left(\int_a^y \cos\ t^2\ dt\right)\frac{dy}{dx} = \cos\ y^2\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2\sin x^2}{x\cos y^2}$$

C-6. If $\int_{\sin x}^{1} t^2\ (f(t))\ dt = (1 - \sin x)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is **[16JM120543]**

यदि $\int_{\sin x}^{1} t^2\ (f(t))\ dt = (1 - \sin x)$ हो, तब $f\left(\frac{1}{\sqrt{3}}\right)$ है–

(A) 1/3 (B) $1/\sqrt{3}$ (C*) 3 (D) $\sqrt{3}$

Sol. Differentiating both sides we get,

$1^2f(1)0 - \sin^2 x\ f(\sin x)\ .\ \cos x = 0 -\ \cos x$

$\Rightarrow \quad f(\sin x) = \frac{1}{\sin^2 x} \Rightarrow \quad f(t) = \frac{1}{t^2} \quad$ therefore $f\left(\frac{1}{\sqrt{3}}\right) = 3$

Hindi दोनों पक्षों का अवकलन करने पर

$1^2f(1)0. - \sin^2 x\ f(\sin x)\ .\ \cos x = 0 -\ \cos x$

$\Rightarrow \quad f(\sin x) = \frac{1}{\sin^2 x} \Rightarrow \quad f(t) = \frac{1}{t^2} \quad$ इस प्रकार $f\left(\frac{1}{\sqrt{3}}\right) = 3$

C-7. The value of $\lim\limits_{a\to\infty} \frac{1}{a^2}\int_0^a \ln(1+e^x)\ dx$ equals

$\lim\limits_{a\to\infty} \frac{1}{a^2}\int_0^a \ln(1+e^x)\ dx$ का मान बराबर है

(A) 0 (B) 1 (C*) $\frac{1}{2}$ (D) non-existent विद्यमान नहीं

Sol. $\lim\limits_{a\to\infty} \frac{\int_0^a \ell n(1+e^x)dx}{a^2} = \lim\limits_{a\to\infty} \int_0^{\pi/2} \frac{\ell n(1+e^a)}{2a}$ (By L'Hopital's Rule) (L-हॉस्पीटल नियम से)

$$= \lim_{a\to\infty} \frac{a+\ell n(1+e^{-a})}{2a} = \frac{1}{2} + \lim_{a\to\infty}\frac{\ell n(1+e^{-a})}{2a} = \frac{1}{2}$$

C-8. $f(x) = \int_{1}^{x} \frac{\sin x \cos y}{y^2+y+1} dy$, then तब

(A) $f'(x) = 0 \ \forall \ x = \frac{n\pi}{2}$, $n \in Z$ (B*) $f'(x) = 0 \ \forall x = (2n+1)\ \frac{\pi}{2}, n \in Z$

(C) $f'(x) = 0 \ \forall \ \ x = n\pi, n \in Z$ (D) $f'(x) \neq 0 \ \forall \ x \in R$

Sol. $f'(x) = \cos x \int_{1}^{x} \frac{\cos y}{y^2+y+1} dy + \sin x . \frac{\cos x}{x^2+x+1}$

Note that दिया गया है $f'(x) = 0 \ \ \forall \ x \in (2n+1)\ \frac{\pi}{2}, n \in Z$

C-9. $\int_{0}^{\pi/2} \sin^4 x \cos^3 x \, dx$ is equal to बराबर है:

(A) $\frac{6}{35}$ (B) $\frac{2}{21}$ (C) $\frac{2}{15}$ (D*) $\frac{2}{35}$

Sol. By Wallis' formula. वॉली सूत्र से $I = \frac{(3\cdot 1)\cdot 2}{7\cdot 5\cdot 3\cdot 1} = \frac{2}{35}$

C-10. $\int_{0}^{1} x^2(1-x)^3 dx$ is equal to बराबर है :

(A*) $\frac{1}{60}$ (B) $\frac{1}{30}$ (C) $\frac{2}{15}$ (D) $\frac{\pi}{120}$

Sol. Put $x = \sin^2\theta$ & apply Wallis formula or solve directly.
$x = \sin^2\theta$ रखने पर तथा वॉली सूत्र लगाने पर हल पर

SECTION (D) : ESTIMATION & MEAN VALUE THEOREM
SECTION (D) : अनुमानित मान तथा माध्य मान प्रमेय

D-1.✍ Let माना $I = \int_{1}^{3} \sqrt{x^4+x^2}\, dx$, then तब

(A) $I > \ 6\sqrt{10}$ (B) $I < \ 2\sqrt{2}$ (C*) $2\sqrt{2} \ < I < \ 6\sqrt{10}$ (D) $I < 1$

Sol. Note that दिया गया है कि $\sqrt{2} < \sqrt{x^4+x^2} < 3\sqrt{10} \ < \forall \ x \in (1, 3)$

Hence अतः $2\sqrt{2} \ < \int_{1}^{3} \sqrt{x^4+x^3} dx < 6\sqrt{10}$

D-2.✍ $I = \int_{0}^{2\pi} e^{\sin^2 x + \sin x + 1} dx$, then तब **[16JM120546]**

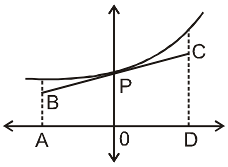
(A) $\pi e^3 < I < 2\pi e^5$ (B*) $2\pi e^{3/4} < I < 2\pi e^3$ (C) $2\pi e^3 < I < 2\pi e^4$ (D) $0 < I < 2\pi$

Sol. $\sin^2 x + \sin x + 1 \ \in \left[\frac{3}{4}, 3\right] \ \forall x \in [0, 2\pi]$

Hence अतः $2\pi e^{3/4} < I < 2\pi e^3$.

D-3.✍ Let $f''(x) \geq 0$, $f'(x) > 0$, $f(0) = 3$ & $f(x)$ is defined in $[-2, 2]$. If $f(x)$ is non-negative, then
माना $f''(x) \geq 0$, $f'(x) > 0$, $f(0) = 3$, $f(x)$, $[-2, 2]$ में परिभाषित है। यदि $f(x)$ अऋणात्मक है तब

(A) $\int_{-1}^{0} f(x)dx > 6$ (B) $\int_{-2}^{2} f(x)dx > 12$ (C*) $\int_{-2}^{2} f(x)dx \geq 12$ (D) $\int_{-1}^{1} f(x)dx > 12$



Sol. $\int_{-2}^{2} f(x)dx \geq \frac{1}{2} \times (AB + CD) \times 4$

$= 2(AB + CD) = 2(2OP) = 12$

& equality holds if f(x) is a linear function.
समता होगी यदि f(x) रैखिक फलन है।

D-4. Let mean value of $f(x) = \frac{1}{x+c}$ over interval (0, 2) is $\frac{1}{2}\ell n3$ then positive value of c is

माना $f(x) = \frac{1}{x+c}$ का अन्तराल (0, 2) पर माध्यमान $\frac{1}{2}\ell n3$ है तब c का धनात्मक मान बराबर है–

(A*) 1 (B) $\frac{1}{2}$ (C) 2 (D) $\frac{3}{2}$

Sol. $\frac{1}{2-0}\int_{0}^{2}\frac{1}{x+c}dx = \frac{1}{2}\ell n3 \Rightarrow \frac{1}{2}\ell n \left|\ln(x+c)\right|_0^2 = \frac{1}{2}\ell n3 \Rightarrow \left|\frac{2+c}{c}\right| = \ell n3 \Rightarrow |2 + c| = 3|c|$

SECTION (E) : INTEGRATION AS A LIMIT OF SUM AND REDUCTION FORMULA
SECTION (E) : सीमाओं के योग से समाकलन तथा समानयन सूत्र

E-1. $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r^3}{r^4+n^4}\right)$ equals to :

$\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{r^3}{r^4+n^4}\right)$ का मान है–

(A) $\ell n\ 2$ (B) $\frac{1}{2}\ell n\ 2$ (C) $\frac{1}{3}\ell n\ 2$ (D*) $\frac{1}{4}\ell n\ 2$

Sol. $I = \lim_{n\to\infty}\sum_{r=1}^{n}\left[\frac{\frac{1}{n}\left(\frac{r^3}{n^3}\right)}{\frac{r^4}{n^4}+1}\right] = \int_{0}^{1}\frac{x^3}{1+x^4}dx = \frac{1}{4}\ell n2$

E-2. $\underset{n\to\infty}{\text{Lt}}\sum_{r=2n+1}^{3n}\frac{n}{r^2-n^2}$ is equal to : **[16JM120547]**

$\underset{n\to\infty}{\text{Lt}}\sum_{r=2n+1}^{3n}\frac{n}{r^2-n^2}$ का मान है–

(A) $\ell n\sqrt{\frac{2}{3}}$ (B*) $\ell n\sqrt{\frac{3}{2}}$ (C) $\ell n\frac{2}{3}$ (D) $\ell n\frac{3}{2}$

Sol. $\lim\limits_{n\to\infty}\sum\limits_{2n+1}^{3n}\frac{n}{r^2-n^2} = \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{2n\ +1}^{3n}\frac{1}{\left(\frac{r}{n}\right)^2-1} = \int_2^3\frac{dx}{x^2-1} = \left[\frac{1}{2}\ell n\left|\frac{x-1}{x+1}\right|\right]_2^3 = \ell n\sqrt{\frac{3}{2}}$

E-3. $\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ ...\ \left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$ is equal to :

(A) $\frac{e^{\pi/2}}{2e^2}$ (B) $2\ e^2\ e^{\pi/2}$ (C*) $\frac{2}{e^2}\ e^{\pi/2}$ (D) $2\ e^{\pi}$

$\lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ ...\ \left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$ का मान है–

(A) $\frac{e^{\pi/2}}{2e^2}$ (B) $2\ e^2\ e^{\pi/2}$ (C) $\frac{2}{e^2}\ e^{\pi/2}$ (D) $2\ e^{\pi}$

Sol. $S = \lim\limits_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ ...\ \left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

$\ell n\ S = \lim\limits_{n\to\infty}\left[\ell n\ \left(1+\frac{1}{n^2}\right)+\ell n\ \left(1+\frac{2^2}{n^2}\right)+........+\ell n\ \left(1+\frac{n^2}{n^2}\right)\right]$

$= \lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{r=1}^{n}\ell n\ \left(1+\frac{r^2}{n^2}\right) = \int_0^1 1.\ell n\ (1+x^2)\ dx \Rightarrow \ell nS = \frac{\pi}{2}-2+\ell n2$

$S = 2e^{\frac{\pi}{2}-2} = 2e^{\frac{\pi}{2}}\cdot e^{-2} = \frac{2}{e^2}\ e^{\frac{\pi}{2}}$

E-4. $\lim\limits_{n\to\infty}\frac{\pi}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+.....+\sin\frac{(n-1)\pi}{n}\right]$ is equals to : **[16JM120548]**

(A) 0 (B) π (C*) 2 (D) 3

$\lim\limits_{n\to\infty}\frac{\pi}{n}\left[\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+.....+\sin\frac{(n-1)\pi}{n}\right]$ का मान है–

(A) 0 (B) π (C) 2 (D) 3

Sol. $I = \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n-1}\frac{\pi}{n}\sin\frac{r\pi}{n} = \pi\int_0^1 \sin\pi x\ \ dx = \pi = [-\cos\pi + 1] = 2$

E-5. Let माना $I_n = \int_0^1 (1-x^3)^n dx,\ (n\in N)$ then तब

(A) $3n\ I_n = (3n-1)\ I_{n-1}\ \forall\ n\ge 2$ (B) $(3n-1)I_n = 3n\ I_{n-1}\forall\ n\ge 2$

(C) $(3n-1)I_n = (3n+1)\ I_{n-1}\ \forall\ n\ge 2$ (D*) $(3n+1)I_n = 3n\ I_{n-1}\ \forall\ n\ge 2$

Sol. $I_n = \int_0^1 (1-x^3)^n.1\,dx = (1-x^3)^n.\ x|_0^1 - 3n\int_0^1 (1-x^3)^{n-1}(1-x^3-1)dx$

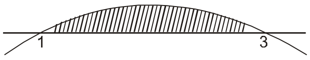
$\Rightarrow\ \ I_n = 0 - 3n(I_n - I_{n-1})\ \ \Rightarrow\ \ (3n+1)\ I_n = 3nI_{n-1}$

Section (F) : Area Under Curve
खण्ड (F) : वक्र से परिबद्ध क्षेत्रफल

F-1. The area bounded by the x-axis and the curve $y = 4x - x^2 - 3$ is
x-अक्ष तथा वक्र $y = 4x - x^2 - 3$ से परिबद्ध क्षेत्रफल है–

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C*) $\frac{4}{3}$ (D) $\frac{8}{3}$

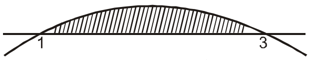
Sol. y = 0 $\Rightarrow$ x = 1, 3



Graph of y = 4x –x²–3

Area = $\int_1^3 (4x - x^2 - 3)dx = \frac{4}{3}$

Hindi y = 0 $\Rightarrow$ x = 1, 3



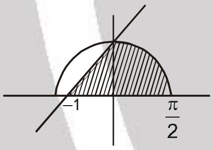
y = 4x –x²–3 का आरेख

क्षेत्रफल $= \int_1^3 (4x - x^2 - 3)dx = \frac{4}{3}$

F-2. The area of the figure bounded by right of the line y = x + 1, y = cos x and x–axis is: **[16JM120549]**
रेखा y = x + 1 के दाँयी ओर, y = cos x एवं x–अक्ष से परिबद्ध आकृति का क्षेत्रफल है–

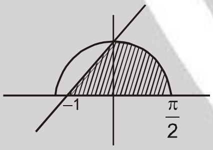
(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{5}{6}$ (D*) $\frac{3}{2}$

Sol. From figure it is clear that required



area = $\frac{1}{2} + \int_0^{\pi/2} \cos x \ dx = \frac{3}{2}$

Hindi चित्र से यह स्पष्ट है कि

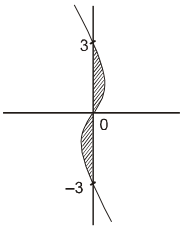


अभीष्ट क्षेत्रफल = = $\frac{1}{2} + \int_0^{\pi/2} \cos x \ dx = \frac{3}{2}$

F-3. Area bounded by curve $y^3 - 9y + x = 0$ and y-axis is
वक्र $y^3 - 9y + x = 0$ तथा y-अक्ष से परिबद्ध क्षेत्रफल है–

(A) $\frac{9}{2}$ (B) 9 (C*) $\frac{81}{2}$ (D) 81

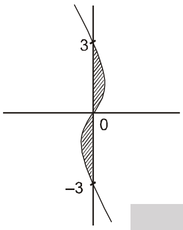
Sol. x = 0 $\Rightarrow$ y = 0, – 3, 3



Figure

Required area = $2\int_0^3 (9y \;\; -y^3) \;\; dy \;\; = \frac{81}{2}$

Hindi. x = 0 ⇒ y = 0, – 3, 3



Figure

अभीष्ट क्षेत्रफल = $2\int_0^3 (9y \;\; -y^3) \;\; dy \;\; = \frac{81}{2}$

F-4. Let f:[0, ∞) → R be a continuous and strictly increasing function such that $f^3(x) = \int_0^x t \;\; f^2(t) \;\; dt$, x ≥ 0.

The area enclosed by y = f(x), the x-axis and the ordinate at x = 3 is ———— **[16JM120550]**

माना f:[0, ∞) → R एक सतत् तथा एकदिष्ट वर्द्धमान फलन इस प्रकार है कि $f^3(x) = \int_0^x t \;\; f^2(t) \;\; dt$, x ≥ 0 तो y = f(x), x-अक्ष तथा x = 3 की कोटि द्वारा परिबद्ध क्षेत्रफल ———— है।

(A*) $\frac{3}{2}$ (B) $\frac{5}{2}$ (C) $\frac{7}{2}$ (D) $\frac{1}{2}$

Sol. Differentiating w.r.t. x (x के सापेक्ष अवकलन करने पर)

$3f^2(x)\,f'(x) = x\,f^2(x)$

$f'(x) = \frac{x}{3}$

$f(x) = \frac{x^2}{6} + C$; Area (क्षेत्रफल) = $\int_0^3 \frac{x^2}{6} dx = 3/2$

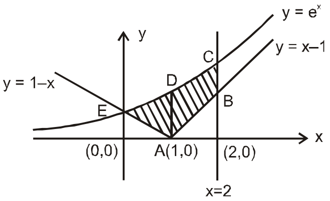
F-5. The area bounded by the curve y = e^x and the lines y = $|x-1|$, x = 2 is given by:

(A) e² + 1 (B) e² – 1 (C*) e² – 2 (D) e – 2

वक्र y = e^x तथा रेखाओं y = $|x-1|$, x = 2 से परिबद्ध क्षेत्रफल दिया जाता है–

(A) e² + 1 से (B) e² – 1 से (C) e² – 2 से (D) e – 2

Sol. Area = $\int_0^1 \left(e^x - (1-x)\right) \;\; dx + \int_1^2 \left(e^x - (x-1)\right) \;\; dx$



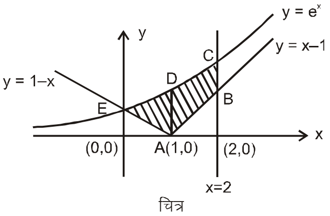
Figure

$$= \left(e^x - x + \frac{x^2}{2}\right)_0^1 + \left(e^x - \frac{x^2}{2} + x\right)_1^2$$

$$= \left(e^1 - 1 + \frac{1}{2}\right) - 1 + \left(e^2 - 2 + 2\right) - \left(e^1 - \frac{1}{2} + 1\right)$$

$$= e^2 - 2$$

Hindi. क्षेत्रफल $= \int_0^1 \left(e^x - (1-x)\right)\ dx + \int_1^2 \left(e^x - (x-1)\right)\ dx$



चित्र

$$= \left(e^x - x + \frac{x^2}{2}\right)_0^1 + \left(e^x - \frac{x^2}{2} + x\right)_1^2 = \left(e^1 - 1 + \frac{1}{2}\right) - 1 + \left(e^2 - 2 + 2\right) - \left(e^1 - \frac{1}{2} + 1\right) = e^2 - 2$$

F-6. The area bounded by $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$ is: **[16JM120551]**

$y = 2 - |2 - x|$ तथा $y = \frac{3}{|x|}$ से परिबद्ध क्षेत्रफल है–

(A) $\frac{4 + 3\ \ell n\ 3}{2}$ (B*) $\frac{4 - 3\ \ell n\ 3}{2}$ (C) $\frac{3}{2} + \ell n\ 3$ (D) $\frac{1}{2} + \ell n\ 3$

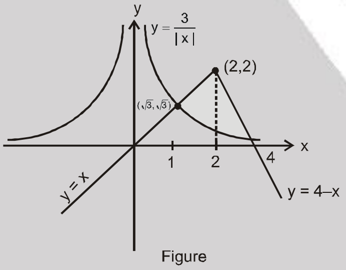
Sol. When जब $x < 2$

$2 - 2 + x = \frac{3}{x} \Rightarrow x = \sqrt{3}$

when जब $x > 2$

$= 2 + 2 - x = \frac{3}{x} \Rightarrow X = 3, 1$

$$A = \int_{\sqrt{3}}^{2} \left(x - \frac{3}{x}\right)\ dx + \int_{2}^{3} \left(4 - x - \frac{3}{x}\right)\ dx$$



Figure

$$= \left(\frac{x^2}{2} - 3\ell nx\right)_{\sqrt{3}}^{2} + \left(4x - \frac{x^2}{2} - 3\ell nx\right)_{2}^{3}$$

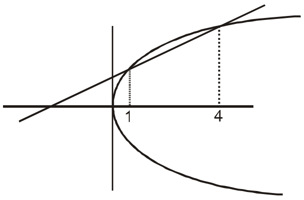
$$= \frac{4 - 3\ell n3}{2}$$

F-7. The area bounded by the curve $y^2 = 4x$ and the line $2x - 3y + 4 = 0$ is
वक्र $y^2 = 4x$ तथा रेखा $2x - 3y + 4 = 0$ से परिबद्ध क्षेत्रफल है–

(A*) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{3}$

Sol. Solving x = 1, 4

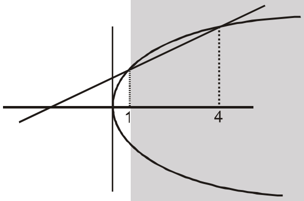
From graph it is clear that required



area = $\int_1^4 \left(2\sqrt{x} - \frac{1}{3}(2x+4)\right)dx = \frac{1}{3}$

Hindi हल करने पर x = 1, 4

आरेख से यह स्पष्ट है कि



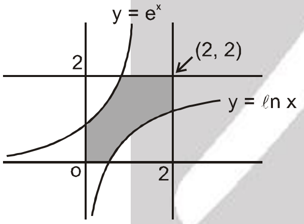
अभीष्ट क्षेत्रफल = $\int_1^4 \left(2\sqrt{x} - \frac{1}{3}(2x+4)\right)dx = \frac{1}{3}$

F-8. The area of the region bounded by x = 0, y = 0, x = 2, y = 2, $y \le e^x$ and $y \ge \ell n\, x$, is **[16JM120552]**

x = 0, y = 0, x = 2, y = 2, $y \le e^x$ तथा $y \ge \ell n\, x$ से परिबद्ध क्षेत्र का क्षेत्रफल है–

(A*) $6 - 4\,\ell n\, 2$ (B) $4\,\ell n\, 2 - 2$ (C) $2\,\ell n\, 2 - 4$ (D) $6 - 2\,\ell n\, 2$

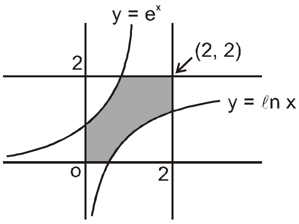
Sol. $A = \int_1^2 \ell n\ x\ dx = 2\,\ell n\, 2 - 1$



Figure

$\Rightarrow$ Required area = $4 - 2\,(2\ell n\, 2 - 1) = 6 - 4\,\ell n\, 2$

Hindi. $A = \int_1^2 \ell n\ x\ dx = 2\,\ell n\, 2 - 1$

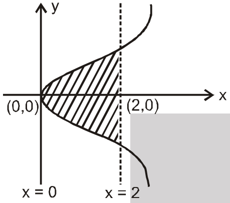


Figure

$\Rightarrow$ अभीष्ट क्षेत्रफल $= 4 - 2\,(2\ell n\, 2 - 1) = 6 - 4\,\ell n\, 2$

F-9. The area between two arms of the curve $|y| = x^3$ from $x = 0$ to $x = 2$ is
वक्र $|y| = x^3$ की दो भुजाओं का $x = 0$ से $x = 2$ के मध्य परिबद्ध क्षेत्र का क्षेत्रफल हैं–

(A) 2 (B) 4 (C*) 8 (D) 16



Sol. $A = 2\int_0^2 x^3 dx = 8$

F-10. The area bounded by the parabolas $y = (x + 1)^2$ and $y = (x - 1)^2$ and the line $y = \frac{1}{4}$ is **[16JM120553]**

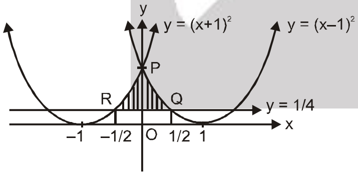
(A) 4 sq. units (B) $\frac{1}{6}$ sq. units (C) $\frac{4}{3}$ sq. units (D*) $\frac{1}{3}$ sq. units

परवलयों $y = (x + 1)^2$ एवं $y = (x - 1)^2$ तथा रेखा $y = \frac{1}{4}$ से परिबद्ध क्षेत्र का क्षेत्रफल है –

(A) 4 वर्ग इकाई (B) $\frac{1}{6}$ वर्ग इकाई (C) $\frac{4}{3}$ वर्ग इकाई (D) $\frac{1}{3}$ वर्ग इकाई

Sol. The curves $y = (x - 1)^2$, $y = (x + 1)^2$ and $y = 1/4$ are shown as :
point of intersection are

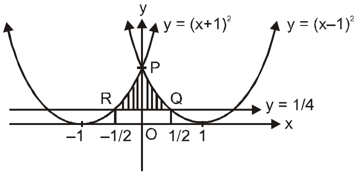
$\therefore$ P(0,1) , $Q = \left(\frac{1}{2}, \frac{1}{4}\right)$ and $R = \left(\frac{-1}{2}, \frac{1}{4}\right)$



$$\text{Area} = 2\int_0^{1/2}\left\{(x-1)^2 - \frac{1}{4}\right\} dx \quad = 2\left\{\frac{(x-1)^3}{3} - \frac{1}{4}x\right\}_0^{1/2} \quad = \frac{1}{3} \text{ sq. units}$$

Hindi. वक्रों $y = (x - 1)^2$, $y = (x + 1)^2$ तथा $y = 1/4$ का आलेख है
प्रतिच्छेद बिन्दु है

$\therefore$ P(0,1) , $Q = \left(\frac{1}{2}, \frac{1}{4}\right)$ तथा $R = \left(\frac{-1}{2}, \frac{1}{4}\right)$



$$\text{क्षेत्रफल} = 2\int_0^{1/2}\left\{(x-1)^2 - \frac{1}{4}\right\} dx$$

$$= 2\left\{\frac{(x-1)^3}{3} - \frac{1}{4}x\right\}_0^{1/2} = \frac{1}{3} \text{ वर्ग इकाई}$$

PART - III : MATCH THE COLUMN

भाग - III : कॉलम को सुमेलित कीजिए (MATCH THE COLUMN)

1. Let $\underset{T\to\infty}{\text{Lim}}\frac{1}{T}\int_0^T(\sin x + \sin ax)^2 dx = L$ then

Column - I	**Column- II**
(A) for a = 0, the value of L is	(p) 0
(B) for a = 1 the value of L is	(q) 1/2
(C) for a = – 1 the value of L is	(r) 3/2
(D) $\forall$ a $\in$ R – {–1, 0, 1} the value of L is	(s) 2
	(t) 1

Hindi. $\underset{T\to\infty}{\text{Lim}}\frac{1}{T}\int_0^T(\sin x + \sin ax)^2 dx = L$ माना तब

Column - I	**Column- II**
(A) a = 0 के लिए L का मान है	(p) 0
(B) a = 1 के लिए L का मान है	(q) 1/2
(C) a = –1 के लिए L का मान है	(r) 3/2
(D) सभी a $\in$ R – {–1, 0, 1} के लिए L का मान है	(s) 2
	(t) 1

Ans. A - q, B - s, C - p, D - t

Sol. Let माना $I = \dfrac{\underset{T\to\infty}{\text{Lim}}\int_0^T(\sin x + \sin ax)^2 dx}{T}$

(A) If यदि a = 0, $I = \lim\limits_{T\to\infty}\dfrac{\int_0^T \sin^2 x\,dx}{T} = \lim\limits_{T\to\infty}\dfrac{\int_0^T (1-\cos 2x)\,dx}{2T} = \lim\limits_{T\to\infty}\dfrac{T - \frac{\sin 2T}{2}}{2T} = \dfrac{1}{2}$

(B) If यदि a = 1, $I = \lim\limits_{T\to\infty}\dfrac{4\int_0^T \sin^2 x\,dx}{T} = 2$

(C) If यदि a = –1 I = 0

(D) If यदि $a \in R - \{-1, 0\}$

$$I = \lim_{T\to\infty} \frac{\int_0^T (\sin^2 x + \sin^2 ax + 2\sin x \sin ax)dx}{T}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left\{\int_0^T (1-\cos 2x)dx + \int_0^T (1-\cos 2ax)dx + \int_0^T \{\cos(a-1)x - \cos(a+1)x\}\,dx\right\}$$

$$= \lim_{T\to\infty} \frac{1}{2T}\left\{2T - \frac{\sin T}{2} - \frac{\sin 2aT}{2a} + \frac{\sin(a-1)T}{a-1} - \frac{\sin(a+1)}{a+1}\right\} = 1$$

2. **Column – I** **Column – II**

[16JM120554]

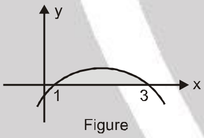
	Column – I		Column – II
(A)	Area bounded by region $0 \le y \le 4x - x^2 - 3$ is	(p)	32/3
(B)	The area of figure formed by all the points satisfying the inequality $y^2 \le 4\,(1 - \lvert x\rvert)$ is	(q)	1/2
(C)	The area bounded by $\lvert x\rvert + \lvert y\rvert \le 1$ and $\lvert x\rvert \ge 1/2$ is	(r)	4/3
(D)	Area bounded by $x \le 4 - y^2$ and $x \ge 0$ is	(s)	16/3

	स्तम्भ – I		स्तम्भ – II
(A)	क्षेत्र $0 \le y \le 4x - x^2 - 3$ से परिबद्ध क्षेत्रफल है–	(p)	32/3
(B)	असमिका $y^2 \le 4\,(1 - \lvert x\rvert)$ को संतुष्ट करने वाले सभी बिन्दुओं से बने क्षेत्र का क्षेत्रफल है–	(q)	1/2
(C)	$\lvert x\rvert + \lvert y\rvert \le 1$ और $\lvert x\rvert \ge 1/2$ से परिबद्ध क्षेत्रफल है–	(r)	4/3
(D)	$x \le 4 - y^2$ एवं $x \ge 0$ से परिबद्ध क्षेत्रफल है–	(s)	16/3

Ans. (A) → (r), (B) → (s), (C) → (q), (D) → (p)

Sol. (A) $0 \le \; y \le 4x - x^2 - 3$

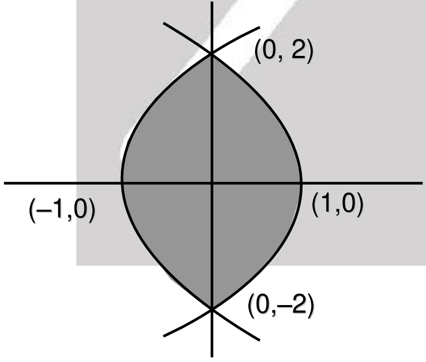
at $y = 0,\ x^2 - 4x + 3 = 0 \Rightarrow x = 1,\ x = 3$



$$A = \int_1^3 (4x - x^2 - 3)\ \ dx = \frac{4}{3}$$

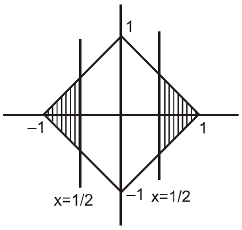
(B) $y^2 \le 4\,(x + 1)$ if $x \le 0$ and $y^2 \le -4\,(x - 1)$ if $x \ge 0$

⇒ (x,y) lies in the shaded region shown in figure



⇒ required area equal to $\int_0^2 (4 - y^2)dy = \left(4y - \frac{y^3}{3}\right)_0^2 = 8 - \frac{8}{3} = \frac{16}{3}$

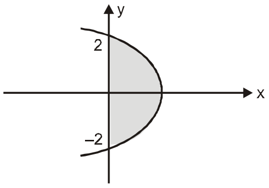
(C)



Figure

$$\text{Required area} = 4\left(\frac{1}{2}\cdot\ \frac{1}{2}\cdot\frac{1}{2}\right) = \frac{1}{2}$$

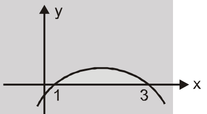
(D)



$$A = \int_{-2}^{2}(4-y^2)\ \ dy = \left(4y-\frac{y^3}{3}\right)_{-2}^{2} = \frac{32}{3}$$

Hindi. (A) $0 \le y \le 4x - x^2 - 3$

$y = 0$ पर $x^2 - 4x + 3 = 0 \quad x = 1,\ x = 3$

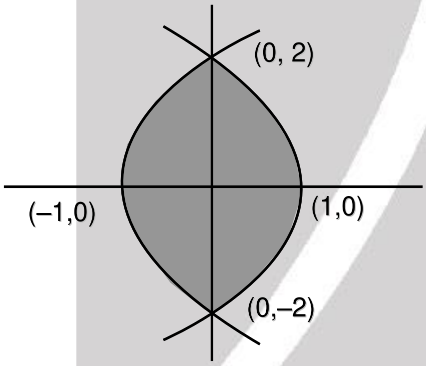


Figure

$$A = \int_{1}^{3}(4x-x^2-3)\ \ dx\ = \ \frac{4}{3}$$

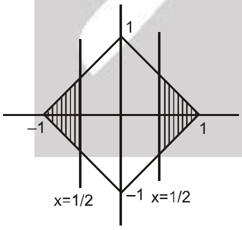
(B) $y^2 \le 4\ (x + 1)$ if $x \le 0$ and और $y^2 \le - 4\ (x - 1)$ if $x \ge 0$

⇒ (x,y) lies in the shaded region shown in figure चित्र में (x,y) छायाकित क्षेत्र में स्थित है



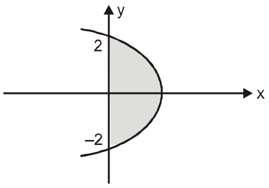
⇒ required area equal to अभीष्ट क्षेत्रफल $\int_{0}^{2}(4-y^2)dy = \left(4y-\frac{y^3}{3}\right)_{0}^{2} = 8 - \frac{8}{3} = \frac{16}{3}$

(C)



चित्र

अभीष्ट क्षेत्रफल $= 4\left(\frac{1}{2}\cdot\ \frac{1}{2}\cdot\frac{1}{2}\right) = \frac{1}{2}$

(D) 

$$A = \int_{-2}^{2} (4 - y^2)\ dy = \left(4y - \frac{y^3}{3}\right)_{-2}^{2} = \frac{32}{3}$$

Exercise-2

✍ **Marked questions are recommended for Revision.**

✍ चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : ONLY ONE OPTION CORRECT TYPE

भाग-I : केवल एक सही विकल्प प्रकार (ONLY ONE OPTION CORRECT TYPE)

1. The value of $\int_0^1 (\{2x\}-1)(\{3x\}-1)\ dx$, (where { } denotes fractional part of x) is equal to :

$\int_0^1 (\{2x\}-1)(\{3x\}-1)\ dx$ का मान है, (जहाँ {x}, x का भिन्नात्मक भाग प्रदर्शित करता है।)

(A) $\frac{19}{36}$ (B) $\frac{19}{144}$ (C*) $\frac{19}{72}$ (D) $\frac{19}{18}$

Sol. $\int_0^1 (\{2x\}-1)(\{3x\}-1)dx$

$$= \int_0^{1/3}(2x-1)\ (3x-1)\ dx + \int_{1/3}^{1/2}(2x-1)\ (3x-2)\ dx + \int_{1/2}^{2/3}(2x-2)\ (3x-2)\ dx + \int_{2/3}^{1}(2x-2)\ \ (3x-3)\ dx$$

$$= \int_0^{1/3}(6x^2-5x+1)\ dx\ +\ \int_{1/3}^{1/2}(6x^2-7x+2)\ \ dx\ +\ \int_{1/2}^{2/3}(6x^2-10x+4)\ dx\ +\ \int_{2/3}^{1}(6x^2-12x+6)\ \ dx\ =\ \frac{19}{72}.$$

2.✍ If $\int_0^{100} f(x)\ \ dx = a$, then $\sum_{r=1}^{100}\left(\int_0^1 f\ \ (r-1+x)\ \ dx\right) =$

यदि $\int_0^{100} f(x)\ \ dx = a$, तब $\sum_{r=1}^{100}\left(\int_0^1 f\ \ (r-1+x)\ \ dx\right)$ का मान है–

(A) 100 a (B*) a (C) 0 (D) 10 a

Sol. $\sum_{r=1}^{100}\left(\int_0^1 f(r-1+x)\ \ dx\right) = \int_0^1 f(x)\ \ dx + \int_0^1 f(1+x)\ \ dx + \int_0^1 f(2+x)\ \ dx + ... + \int_0^1 f(99+x)\ \ dx$

$$= \int_0^1 f(x)\ \ dx + \int_1^2 f(x)\ \ dx + ... + \int_{99}^{100} f(x)\ \ dx$$

{using shifting property} {स्थानांतरण प्रगुण से}

$$= \int_0^{100}\ \ f(x)\ \ dx\ = a$$

3. $\lim_{t\to\left(\frac{\pi}{2}\right)^-}\int_0^t \tan\theta\sqrt{\cos\theta}\ \ell n(\cos\theta)d\theta$ is equal to बराबर है :

(A*) –4 (B) 4 (C) –2 (D) Does not exists विद्यमान नही है

Sol. $\lim_{t\to\left(\frac{\pi}{2}\right)^-}\int_0^t \frac{\sin\theta}{\sqrt{\cos\theta}}.\ln(\cos\theta)d\theta$, let cosθ = y²

$$2\ \lim_{a\to0^+}\int_0^t \frac{\ell n(y^2)}{y}tdy = 4.\ \lim_{a\to0^+}\int_0^t \ell n y dy\ =\ 4\{y\ell ny - y\}_a^1\ =\ 4.\ \lim_{a\to0^+}\{-1+a-a\ell na\}\ = -4$$

4. If $f(x) = \begin{cases} 0\ , & \text{where } \ x = \dfrac{n}{n+1},\ n = 1,\ 2,\ 3..... \\ 1\ , & \text{else where} \end{cases}$, then the value of $\int_0^2 f(x)\ dx$. **[16JM120556]**

यदि $f(x) = \begin{cases} 0\ , & \text{जहाँ } \ x = \dfrac{n}{n+1},\ n = 1,\ 2,\ 3..... \\ 1\ , & \text{अन्यथा} \end{cases}$ हो, तो $\int_0^2 f(x)\ dx$ का मान है–

(A) 1 (B) 0 (C*) 2 (D) ∞

Sol. $\int_0^2 f(x)\ dx = \int_0^{1/2} 1.dx + \int_{1/2}^{2/3} 1.dx + \int_{2/3}^{3/4} 1.dx + ... + \int_{\frac{n-1}{n}}^{\frac{n}{n+1}} 1.dx\ + ...+ \int_1^2 1.\ dx$

$= \left(\frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{2}{3}\right) + ... + \left(\frac{n}{n+1} - \frac{n-1}{n}\right) + ... + 1 = \frac{n}{n+1} + ... + 1$ as $n \to \infty$

taking limit $n \to \infty$

we get $\int_0^2 f(x)\ dx = 1 + 1 = 2$

Hindi $\int_0^2 f(x)\ dx = \int_0^{1/2} 1.dx + \int_{1/2}^{2/3} 1.dx + \int_{2/3}^{3/4} 1.dx + ... + \int_{\frac{n-1}{n}}^{\frac{n}{n+1}} 1.dx\ + ...+ \int_1^2 1.\ dx$

$= \left(\frac{1}{2}\right) + \left(\frac{2}{3} - \frac{1}{2}\right) + \left(\frac{3}{4} - \frac{2}{3}\right) + ... + \left(\frac{n}{n+1} - \frac{n-1}{n}\right) + ... + 1 = \frac{n}{n+1} + ... + 1$ as $n \to \infty$

जैसे $n \to \infty$

$n \to \infty$ लेने पर $\int_0^2 f(x)\ dx = 1 + 1 = 2$

5. If $\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$, then $\int_0^\infty e^{-ax^2}\ dx$ where $a > 0$ is :

यदि $\int_0^\infty e^{-x^2}\ dx = \frac{\sqrt{\pi}}{2}$ हो तो $\int_0^\infty e^{-ax^2}\ dx$, जहाँ $a > 0$, है–

(A) $\frac{\sqrt{\pi}}{2}$ (B) $\frac{\sqrt{\pi}}{2a}$ (C) $2\frac{\sqrt{\pi}}{a}$ (D*) $\frac{1}{2}\sqrt{\frac{\pi}{a}}$

Sol. Let (माना) $I = \int_0^\infty e^{-ax^2} dx$

Put $\sqrt{ax} = t$ (रखने पर) $\Rightarrow$ $dx = \frac{dt}{\sqrt{a}}$

then (तब) $I = \frac{1}{\sqrt{a}} \int_0^\infty e^{-t^2} dt = \frac{1}{\sqrt{a}} \int_0^\infty e^{-x^2} dx = \frac{1}{\sqrt{a}}\ \frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

6. If $\sum_{i=1}^{4}\left(\sin^{-1} x_i + \cos^{-1} y_i\right) = 6\pi$, then $\int_{\sum_{i=1}^{4} x_i}^{\sum_{i=1}^{4} y_i} x\ln(1+x^2)\left(\frac{e^x}{1+e^{2x}}\right)$ dx is equal to **[16JM120557]**

यदि $\sum_{i=1}^{4}\left(\sin^{-1}x_i+\cos^{-1}y_i\right) = 6\pi$, तब $\int\limits_{\sum_{i=1}^{4}x_i}^{\sum_{i=1}^{4}y_i} x\ln(1+x^2)\left(\frac{e^x}{1+e^{2x}}\right)$ dx बराबर है

(A*) 0 (B) $e^4 + e^{-4}$ (C) $\ln\left(\frac{17}{12}\right)$ (D) $e^4 - e^{-4}$

Sol. $\sum_{i=1}^{4}\left(\sin^{-1}x_i+\cos^{-1}y_i\right) = 6\pi \quad \Rightarrow \quad x_1 = x_2 = x_3 = x_4 = 1 \quad \& \quad y_1 = y_2 = y_3 = y_4 = -1$

hence अतः $I = \int_{-4}^{4} x\ln(1+x^2)\left(\frac{e^x}{1+e^{2x}}\right)dx = 0$ (as f(x) is an odd function) (चूंकि f(x) विषम फलन है)

7. The tangent to the graph of the function y = f(x) at the point with abscissa x =1 form an angle of $\pi/6$ and at the point x = 2, an angle of $\pi/3$ and at the point x = 3, an angle of $\pi/4$ with positive x-axis. The value of $\int_1^3 f'(x)\ f''(x)\ dx + \int_2^3 f''(x)\ dx$ (f"(x) is supposed to be continuous) is :

(A) $\frac{4\sqrt{3}-1}{3\sqrt{3}}$ (B) $\frac{3\sqrt{3}-1}{2}$ (C) $\frac{4-\sqrt{3}}{3}$ (D*) $\frac{4}{3}-\sqrt{3}$

फलन y = f(x) के आरेख पर भुज x =1 पर स्पर्श रेखा $\pi/6$ कोण बनाती है। तथा x = 2, पर $\pi/3$ कोण बनाती है, तथा x = 3 पर धनात्मक x- अक्ष के साथ $\pi/4$ कोण बनाती है। तब $\int_1^3 f'(x)\ f''(x)\ dx + \int_2^3 f''(x)\ dx$ मान है। (माना कि f"(x) सतत् है।) है।

(A) $\frac{4\sqrt{3}-1}{3\sqrt{3}}$ (B) $\frac{3\sqrt{3}-1}{2}$ (C) $\frac{4-\sqrt{3}}{3}$ (D*) $\frac{4}{3}-\sqrt{3}$

Sol. Given, (दिया है)

$$f'(1) = \left(\frac{dy}{dx}\right)_{x=1} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$f'(2) = \left(\frac{dy}{dx}\right)_{x=2} = \tan\frac{\pi}{3} = \sqrt{3}$$

$$f'(3) = \left(\frac{dy}{dx}\right)_{x=3} = \tan\frac{\pi}{4} = 1$$

Let, (माना) $I = \int_1^3 f'(x).\ f''(x)dx + \int_2^3 f''(x)dx = I_1 + I_2$

$$\therefore \quad I_1 = \int_1^3 f'(x).\ f''(x)dx$$

$$I_1 = f'(x).\ f'(x)\Big|_1^3 - \int_1^3 f''(x)\ f'(x)\ dx$$

$$2I_1 = \{f'(3)\}^2 - \{f'(1)\}^2$$

$$2I_1 = 1 - \frac{1}{3}$$

$$I_1 = \frac{1}{3}$$

and, (तथा) $\quad I_2 = \int_2^3 f''(x)dx = f'(x)\Big|_2^3 = f'(3) - f'(2) = 1 - \sqrt{3}$

$\therefore \quad I = I_1 + I_2 = \frac{1}{3} + 1 - \sqrt{3} = \frac{4}{3} - \sqrt{3}$

8.✍ Let $A = \int_0^1 \frac{e^t}{1+t}\,dt$, then $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\,dt$ has the value : **[16JM120558]**

माना $A = \int_0^1 \frac{e^t}{1+t}\,dt$ हो, तो $\int_{a-1}^{a} \frac{e^{-t}}{t-a-1}\,dt$ का मान है–

(A) Ae^{-a} (B*) $-Ae^{-a}$ (C) $-ae^{-a}$ (D) Ae^{a}

Sol. $A = \int_0^1 \frac{e^t dt}{t+1}$ $\quad\therefore\quad I = \int_{a-1}^{a} \frac{e^{-t}dt}{t-a-1}$

Put $t = a - y$ (रखने पर) $\Rightarrow$ $dt = -dy$

then (तब) $\quad I = -e^{-a} \int_0^1 \frac{e^y dy}{y+1} = -A\ e^{-a}$

9. $\int_1^2 x^{2x^2+1}(1+2\ell nx)dx$ is equal to बराबर है

(A) 256 (B) 255 (C*) $\frac{255}{2}$ (D) 128

Sol. Substitute रखने पर $\Rightarrow$ x^{x^2} $(x + 2x\ell nx).dx = dt$ $\Rightarrow$ x^{x^2+1} $(1 + 2\ell nx).dx = dt$

Hence अत: $I = \int_1^{16} tdt = \frac{t^2}{2}\Big|_1^{16} = \frac{255}{2}$

10. If f(x) is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x, then $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\ dx$ equals to :

(A) $\sin\theta + \operatorname{cosec}\theta$ (B) $\sin^2\theta$ (C) $\operatorname{cosec}^2\theta$ (D*) none of these

यदि f(x) एक फलन है जो $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ को सभी अशून्य x के लिए संतुष्ट करता है, तो $\int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\ dx$ का मान है–

(A) $\sin\theta + \operatorname{cosec}\theta$ (B) $\sin^2\theta$ (C) $\operatorname{cosec}^2\theta$ (D) इनमें से कोई नहीं

Sol. $f\left(\frac{1}{x}\right) + x^2 f(x) = 0 \quad\Rightarrow\quad f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right)$

$\Rightarrow \quad I = \int_{\sin\theta}^{\operatorname{cosec}\theta} f(x)\ dx = \int_{\sin\theta}^{\operatorname{cosec}\theta} -\frac{1}{x^2} f\left(\frac{1}{x}\right) dx$

$\frac{1}{x} = t$ रखने पर $\Rightarrow \quad -\frac{1}{x^2}dx = dt$

$\Rightarrow \quad I = \int_{\operatorname{cosec}\theta}^{\sin\theta} f(t)\ dt \Rightarrow \quad I = -\int_{\sin\theta}^{\operatorname{cosec}\theta} f(t)\ dt = -I \Rightarrow \quad 2I = 0 \Rightarrow \quad I = 0$

11. If $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}=0$, where C_0, C_1, C_2 are all real, the equation $C_2x^2 + C_1x + C_0 = 0$ has:

(A*) atleast one root in (0, 1)
(B) one root in (1, 2) & other in (3, 4)
(C) one root in (–1, 1) & the other in (–5, –2)
(D) both roots imaginary

यदि $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}=0$, जहाँ C_0, C_1, C_2 सभी वास्तविक है, तब समीकरण $C_2x^2 + C_1x + C_0 = 0$ का है –

(A*) (0, 1) में कम से कम एक मूल
(B) (1, 2) में पहला मूल तथा (3, 4) में दूसरा मूल
(C) (–1, 1) में प्रथम मूल तथा (–5, –2) में दूसरा मूल
(D) दोनों मूल काल्पनिक

Sol. $\int_0^1 (C_2x^2 + C_1x + C_0)\,dx$

$$= \left[\frac{C_2x^3}{3}+\frac{C_1x^2}{2}+C_0x\right]_0^1 \quad = \frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3} \quad = 0 \text{ (given)}$$

⇒ graph $y = C_2x^2 + C_1x + C_0$ crosses x-axis atleast once.

⇒ at least one root of the equation $C_2x^2 + C_1x + C_0 = 0$ is present in (0, 1)

Hindi $\int_0^1 (C_2x^2 + C_1x + C_0)\,dx$

$$= \left[\frac{C_2x^3}{3}+\frac{C_1x^2}{2}+C_0x\right]_0^1 \quad = \frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3} \quad = 0$$ (दिया गया है)

⇒ $y = C_2x^2 + C_1x + C_0$ का आलेख x-अक्ष को कम से कम एक बार प्रतिच्छेद करता है।

⇒ समीकरण $C_2x^2 + C_1x + C_0 = 0$ का कम से कम एक मूल अन्तराल (0, 1) में उपस्थित है।

12. If f(x) = $\int_0^x (2\cos^2 3t + 3\sin^2 3t)\,dt$, f(x + π) is equal to :

(A) f(x) + 2f(π)
(B*) f(x) + 2f $\left(\frac{\pi}{2}\right)$
(C) f(x) + 4f $\left(\frac{\pi}{4}\right)$
(D) 2f(x)

यदि f(x) = $\int_0^x (2\cos^2 3t + 3\sin^2 3t)\,dt$ हो, तो f(x + π) का मान है–

(A) f(x) + 2f(π)
(B) f(x) + 2f $\left(\frac{\pi}{2}\right)$
(C) f(x) + 4f $\left(\frac{\pi}{4}\right)$
(D) 2f(x)

Sol. $f(x+\pi) = \int_0^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)\,dt = \int_0^{x} (2\cos^2 3t + 3\sin^2 3t)\,dt + \int_x^{x+\pi} (2\cos^2 3t + 3\sin^2 3t)\,dt$

t = x + y

$= f(x) + \int_0^{\pi} (2\cos^2 3y + 3\sin^2 3y)\,dy = f(x) + 2\int_0^{\pi/2} (2\cos^2 3y + 3\sin^2 3y)\,dy = f(x) + 2f\left(\frac{\pi}{2}\right)$

13.✍ Let f (x) = $\int_0^x \frac{dt}{\sqrt{1+t^3}}$ and g (x) be the inverse of f (x), then which one of the following holds good?

माना f (x) = $\int_0^x \frac{dt}{\sqrt{1+t^3}}$ और g (x), f (x) का प्रतिलोम फलन है तब निम्न मे से कौनसा सही है ? **[16JM120559]**

(A) 2g'' = g^2
(B*) 2g'' = $3g^2$
(C) 3g'' = $2g^2$
(D) 3g'' = g^2

Sol. $f'(x) = \frac{1}{\sqrt{1+x^3}}$ & $f''(x) = \frac{-3x^2}{2(1+x^3)^{3/2}}$, Also तथा g(f(x)) = x

$\Rightarrow \quad g''(f(x)) = -\dfrac{f''(x)}{(f'(x))^3} = \dfrac{3x^2}{2(1+x^3)^{3/2}}.(1+x^3)^{3/2} \quad \Rightarrow \quad g''(f(x)) = \dfrac{3x^2}{2}$

Let माना $f(x) = t \quad \Rightarrow \quad g(t) = x$

so इसलिए $2g'' = 3g^2$.

14. Let f(x) is differentiable function satisfying $2\int_1^2 f(tx)dt = x + 2, \ \forall \ x \in R$ Then $\int_0^1 (8f(8x) - f(x) - 21x) \, dx$ equals to

माना f(x) अवकलनीय फलन इस प्रकार है कि $2\int_1^2 f(tx)dt = x + 2, \ \forall \ x \in R$ तब $\int_0^1 (8f(8x) - f(x) - 21x) \, dx$ बराबर है–

(A) 3 (B) 5 (C*) 7 (D) 9

Sol. $\dfrac{2}{x}\int_x^{2x} f(y)dy = x + 2 \quad \Rightarrow \quad 2\int_x^{2x} f(y)dy = x^2 + 2x$

differentiate अवकलन करने पर

$\Rightarrow 2[2f(2x) - f(x)] = 2x + 2 \quad \Rightarrow \quad 2f(2x) - f(x) = x + 1 \quad$ ....(i)

$\Rightarrow 4f(4x) - 2f(2x) = 2\,(2x + 1) = 4x + 2 \quad$ ....(ii)

$\Rightarrow 8f\,(8x) - 4f(4x) = 2(8x + 2) = 16x + 4 \quad$ ....(iii)

Add (i) + (ii) and (iii) we get

(i) + (ii) और (iii) को जोड़ने पर

$\Rightarrow 8f(8x) - f(x) = 21x + 7$

$\Rightarrow \int_0^1 (8f(8x) - f(x) - 21x)dx = \int_0^1 7dx = 7$

15. Let माना $I_n = \int_0^1 x^n(\tan^{-1}x)dx$, $n \in N$, then तब

(A) $(n + 1)I_n + (n - 1)I_{n-2} = \dfrac{\pi}{4} + \dfrac{1}{n} \quad \forall \quad n \geq 3$

(B*) $(n + 1)I_n + (n - 1)I_{n-2} = \dfrac{\pi}{2} - \dfrac{1}{n} \quad \forall \quad n \geq 3$

(C) $(n + 1)I_n - (n - 1)I_{n-2} = \dfrac{\pi}{4} + \dfrac{1}{n} \quad \forall \quad n \geq 3$

(D) $(n + 1)I_n - (n - 1)I_{n-2} = \dfrac{\pi}{2} - \dfrac{1}{n} \quad \forall \quad n \geq 3$

Sol. $I_n = \left.\dfrac{(\tan^{-1}x).x^{n+1}}{n+1}\right|_0^1 - \dfrac{1}{n+1}\int_0^1 \dfrac{x^{n+1}}{1+x^2}dx \quad \Rightarrow \quad (n + 1)I_n = \dfrac{\pi}{4} - \int_0^1 \dfrac{x^{n+1}}{x^2+1}dx$

Similary इसी प्रकार $(n - 1)\ I_{n-2} = \dfrac{\pi}{4} - \int_0^1 \dfrac{x^{n-1}}{x^2+1}dx \Rightarrow (n + 1)I_n + (n - 1)I_{n-2} = \dfrac{\pi}{2} - \int_0^1 \dfrac{x^{n+1}(x^2+1)}{x^2+1}dx = \dfrac{\pi}{2} - \dfrac{1}{n}$

16.✍ If , $u_n = \int_0^{\pi/2} x^n \sin x\,dx$, then the value of $u_{10} + 90\ u_8$ is : **[16JM120560]**

यदि $u_n = \int_0^{\pi/2} x^n \sin x\,dx$, तो $u_{10} + 90\ u_8$ का मान है–

(A) $9\left(\frac{\pi}{2}\right)^8$ (B) $\left(\frac{\pi}{2}\right)^9$ (C*) $10\left(\frac{\pi}{2}\right)^9$ (D) $9\left(\frac{\pi}{2}\right)^9$

Sol. $u_{10} = \int_0^{\pi/2} x^{10}\ .\ \sin x\ \ dx\ =\ -\ (x^{10}\ .\ \cos x)_0^{\pi/2} - \int_0^{\pi/2} 10\ .\ x^9(-\cos x)\ \ dx\ = 10\int_0^{\pi/2} x^9 \cos x\ \ dx$

$= \left(10\ .\ x^9 \sin x\right)_0^{\pi/2} - 10\int_0^{\pi/2} 9.x^8 \sin x\ \ dx$

$u_{10} = 10\ .\left(\frac{\pi}{2}\right)^9 - 90\ .\ u_8\ \Rightarrow u_{10} + 90u_8 = 10\left(\frac{\pi}{2}\right)^9$

17.✍ The value of $\int_{1/e}^{\tan x}\frac{t}{1+t^2}\ dt\ + \int_{1/e}^{\cot x}\frac{1}{t(1+t^2)}dt$, where x ∈ (π/6, π/3), is equal to :

(A) 0 (B) 2 (C*) 1 (D) cannot be determined

$\int_{1/e}^{\tan x}\frac{t}{1+t^2}\ dt\ + \int_{1/e}^{\cot x}\frac{1}{t(1+t^2)}dt$, जहाँ x ∈ (π/6, π/3) का मान है–

(A) 0 (B) 2 (C*) 1 (D) गणना नहीं की जा सकती।

Sol. $I = \int_{1/e}^{\tan x}\frac{t}{1+t^2}\ dt + \int_{1/e}^{\cot x}\frac{1}{t(1+t^2)}\ dt$

put $t = \frac{1}{x}$ रखने पर

$I = \int_{1/e}^{\tan x}\frac{t}{1+t^2}\ dt\ + \int_{e}^{\tan x}\frac{1}{\frac{1}{x}\left(1+\frac{1}{x^2}\right)}\ .\ dx$

$I = \int_{1/e}^{\tan x}\frac{t}{1+t^2} + \int_{\tan x}^{e}\frac{x}{1+x^2}\ dx\ = \int_{1/e}^{e}\frac{t}{1+t^2}\ dt\ = \left(\frac{1}{2}\ell n\ (1+t^2)\right)_{1/e}^{e} = 1$

18. Let A₁ = $\int_0^x\left(\int_0^u f(t)\ dt\right)$ du and A₂ = $\int_0^x$ f(u).(x – u) du then $\frac{A_1}{A_2}$ is equal to :

माना A₁ = $\int_0^x\left(\int_0^u f(t)\ dt\right)$ du और A₂ = $\int_0^x$ f(u).(x – u)du तब $\frac{A_1}{A_2}$ बराबर है–

(A) $\frac{1}{2}$ **(B*)** 1 **(C)** 2 **(D)** –1

Sol. L.H.S. वामपक्ष = $\int_0^x\left\{\int_0^u f(t)\ \ dt\right\}\ du$

Integrating by parts taking 1 as 2nd function

खंडशः समाकलन से 1 को द्वितीय फलन लेने पर

L.H.S.वामपक्ष = $\left[u\int_0^u f(t)\ \ dt\right]_0^x - \int_0^x f(u)\ \ .udu\ = x\int_0^x f(t)\ \ dt - \int_0^x f(u)\ \ .\ \ udu$ =

$x\int_0^x f(u)\ \ du - \int_0^x f(u)\ \ .\ \ udu\ = \int_0^x f(u)\ \ .\ \ (x-u)\ \ du$ = R.H.S. दक्षिणपक्ष

19. $\lim\limits_{n\to\infty}\left(\sin\frac{\pi}{2n}\ .\ \sin\frac{2\pi}{2n}\ .\ \sin\frac{3\pi}{2n}.......\sin\frac{(n-1)\pi}{n}\right)^{1/n}$ is equal to : **[16JM120561]**

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C*) $\frac{1}{4}$ (D) $\frac{3}{4}$

$\lim\limits_{n\to\infty}\left(\sin\frac{\pi}{2n}\ .\ \sin\frac{2\pi}{2n}\ .\ \sin\frac{3\pi}{2n}.......\sin\frac{(n-1)\pi}{n}\right)^{1/n}$ का मान है–

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C*) $\frac{1}{4}$ (D) $\frac{3}{4}$

Sol. $A = \left[\lim\limits_{n\to\infty}\left(\sin\frac{\pi}{2n}\ \sin\frac{2\pi}{2n}.......\sin\frac{2(n-1)}{2n}\right)\right]^{1/n}$

$\Rightarrow \quad \ell nA = \frac{1}{n}\sum\limits_{r=1}^{2(n-1)} \ell\ n\ \sin\frac{r\ \ \pi}{2\ \ n} = \int_0^2 \ell n\ \sin\left(\frac{\pi x}{2}\right)dx$

put $\frac{\pi\ \ x}{2} = t \quad \Rightarrow \quad \ell nA = \frac{2}{\pi}\int_0^{\pi} \ell n\ (\sin\ t)\ dt = \frac{4}{\pi}\int\limits_0^{\pi/2} \ell n\ (\sin\ t)\ dt$

$\Rightarrow \quad \ell n\ A = -2\ \ell n\ 2 \quad \Rightarrow \quad A = \frac{1}{4}$

Hindi $A = \left[\lim\limits_{n\to\infty}\left(\sin\frac{\pi}{2n}\ \sin\frac{2\pi}{2n}.......\sin\frac{2(n-1)}{2n}\right)\right]^{1/n}$

$\Rightarrow \quad \ell nA = \frac{1}{n}\sum\limits_{r=1}^{2(n-1)} \ell\ n\ \sin\frac{r\ \ \pi}{2\ \ n} = \int_0^2 \ell n\ \sin\left(\frac{\pi x}{2}\right)dx$

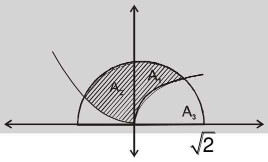
$\frac{\pi\ \ x}{2} = t$ रखने पर $\quad \Rightarrow \quad \ell nA = \frac{2}{\pi}\int_0^{\pi} \ell n\ (\sin\ t)\ dt = \frac{4}{\pi}\int\limits_0^{\pi/2} \ell n\ (\sin\ t)\ dt$

$\Rightarrow \quad \ell n\ A = -2\ \ell n\ 2 \quad \Rightarrow \quad A = \frac{1}{4}$

20. Area bounded by the region consisting of points (x, y) satisfying $y \le \sqrt{2-x^2}$, $y^2 \ge x$, $\sqrt{y} \ge -x$ is

$y \le \sqrt{2-x^2}$, $y^2 \ge x$, और $\sqrt{y} \ge -x$ को संतुष्ठ करने वाले (x, y) बिन्दुओं को रखने वाले क्षेत्र से परिबद्ध क्षेत्रफल है।

(A*) $\frac{\pi}{2}$ (B) π (C) 2π (D) $\pi/4$



Sol. Note that दिया गया है $A_2 = A_3$

Hence अतः $A_1 + A_3 = \frac{\pi}{2}$

21. The area enclosed between the curves **[16JM120562]**

$y = \log_e(x + e)$, $x = \log_e\left(\frac{1}{y}\right)$ and the x-axis is

(A*) 2 (B) 1 (C) 4 (D) 3

वक्रों $y = \log_e(x + e)$, $x = \log_e\left(\frac{1}{y}\right)$ तथा x-अक्ष से परिबद्ध क्षेत्रफल है –

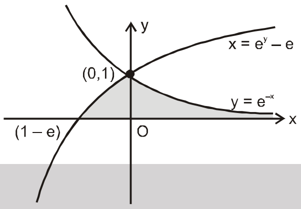
(A) 2 (B) 1 (C) 4 (D) 3

Sol. $y = \log_e(x + e)$, at $y = 0$, $x = 1 - e$ point $(1 - e, 0)$

$e^y = x + e \quad \Rightarrow \quad e^y - e = x$

$x = \log_e\left(\frac{1}{y}\right)$, at $y = e$, $x = -1$

$e^x = \frac{1}{y} \qquad\qquad y = e^{-x}$



Figure

$$\text{Area} = \int_{1-e}^{0} \log_e(x+e)\ dx + \int_{0}^{\infty} e^{-x}\ dx$$

$$= \int_{1}^{e} \ell nt\ \ dt + \int_{0}^{\infty} e^{-x}\ dx = (t\ell nt - t)_1^e + \left(\frac{e^{-x}}{-1}\right)_0^{\infty}$$

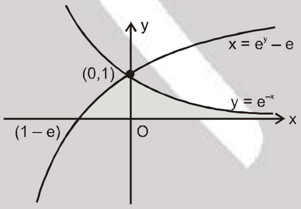
$$= [(e - e) - (0 - 1)] + [-e^{-\infty} + 1] = 1 + 1 = 2$$

Hindi $y = \log_e(x + e)$, $y = 0$ पर, $x = 1 - e$ बिन्दु $(1 - e, 0)$

$e^y = x + e \quad \Rightarrow \quad e^y - e = x$

$x = \log_e\left(\frac{1}{y}\right)$, $y = e$ पर, $x = -1$

$e^x = \frac{1}{y} \qquad\qquad y = e^{-x}$



Figure

$$\text{क्षेत्रफल} = \int_{1-e}^{0} \log_e(x+e)\ dx + \int_{0}^{\infty} e^{-x}\ dx$$

$$= \int_{1}^{e} \ell nt\ \ dt + \int_{0}^{\infty} e^{-x}\ dx = (t\ell nt - t)_1^e + \left(\frac{e^{-x}}{-1}\right)_0^{\infty}$$

$$= [(e - e) - (0 - 1)] + [-e^{-\infty} + 1] = 1 + 1 = 2$$

22. The area bounded by the curve $x = a\cos^3 t$, $y = a \sin^3 t$ is
वक्र $x = a\cos^3 t$, $y = a \sin^3 t$ से परिबद्ध क्षेत्र का क्षेत्रफल है –

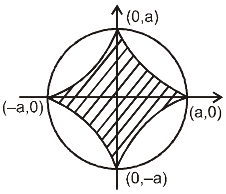
(A*) $\frac{3\pi a^2}{8}$ (B) $\frac{3\pi a^2}{16}$ (C) $\frac{3\pi a^2}{32}$ (D) $3\pi a^2$

Sol. $x = a\cos^3 t$, $y = a\sin^3 t \quad \Rightarrow \quad x^{2/3} + y^{2/3} = a^{2/3}$

$$A = 4\int_0^{\pi/2} y\frac{dx}{dt}dt = 4\int_0^{\pi/2} 3a^2 \sin^3 t \cos^2 t(-\sin t)dt$$

$$= \left|-12a^2 \int_0^{\pi/2} \sin^4 t \quad \cos^2 t \quad dt\right| = \left|-12a^2 \frac{3.1.1}{6.4.2.1} \times \frac{\pi}{2}\right|$$

$$= \frac{3}{8}\pi a^2$$ sq. units वर्ग इकाई



Figure

23. The area bounded by the curve f(x) = x + sin x and its inverse function between the ordinates x = 0 and x = 2π is **[16JM120563]**

वक्र f(x) = x + sin x तथा इसके प्रतिलोम फलन द्वारा कोटियों x = 0 और x = 2π के मध्य परिबद्ध क्षेत्रफल हैं–

(A) 4π (B) 8π (C) 4 (D*) 8

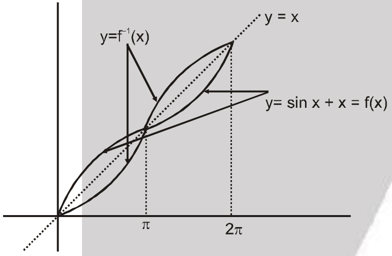
Sol. $\frac{dy}{dx} = 1 + \cos x \geq 0 \ \forall \ x$

$\frac{d^2y}{dx^2} = -\sin x$

x = nπ are points of inflection, where curve changes its concavity.

For x ∈ (0, π), sin x > 0 ⇒ x + sin x > x.

For x ∈ (π, 2π), sin x < 0 ⇒ x + sin x < x.



Figure

Required area = 4A

$$A = \int_0^{\pi} ((x + \sin x) - \quad x)dx$$

A = 2

Required area = 4A = 8

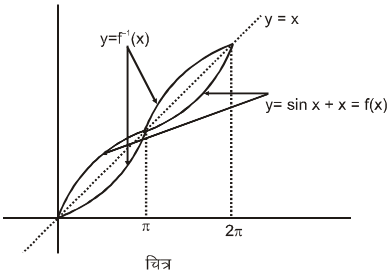
Sol. $\frac{dy}{dx} = 1 + \cos x \geq 0, x$

$\frac{d^2y}{dx^2} = -\sin x$

x = nπ नति परिवर्तन बिन्दु हैं जहाँ वक्र उसकी अवतलता बदलता है।

x ∈ (0, π) के लिए, sin x > 0 ⇒ x + sin x > x

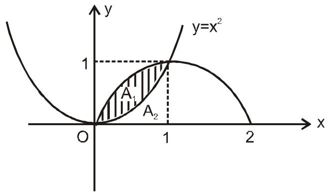
x ∈ (π, 2π) के लिए, sin x < 0 ⇒ x + sin x < x



चित्र

अभीष्ट क्षेत्रफल = 4A

Hindi. $A_1 = \int_0^1 \left(\sin\frac{\pi}{2}x - x^2\right) dx = \left(\frac{-\cos\frac{\pi}{2}x}{\pi/2} - \frac{x^3}{3}\right)_0^1 = \frac{2}{\pi} - \frac{1}{3} = \frac{6-\pi}{3\pi}$



Figure

$A_2 = \int_0^1 x^2 dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$

$\therefore \frac{A_1}{A_2} = \frac{6-\pi/3\pi}{1/3} = \frac{6-\pi}{\pi}$

26. If f(x) = sin x, $\forall\, x \in \left[0, \frac{\pi}{2}\right]$, $f(x) + f(\pi - x) = 2.\ \forall\, x \in \left(\frac{\pi}{2}, \pi\right]$ and $f(x) = f(2\pi - x)$, $x \in (\pi, 2\pi]$, then the area enclosed by y = f(x) and x-axis is

यदि $f(x) = \sin x$, $\forall\, x \in \left[0, \frac{\pi}{2}\right]$, $f(x) + f(\pi - x) = 2.\ \forall\, x \in \left(\frac{\pi}{2}, \pi\right]$ और $f(x) = f(2\pi - x)$, $\forall\, x \in (\pi, 2\pi]$, तब y = f(x) और x-अक्ष से परिबद्ध क्षेत्रफल है

(A) π (B*) 2π (C) 2 (D) 4

Sol. $f(x) + f(\pi - x) = 2.\ \forall\ x \in \left(\frac{\pi}{2}, \pi\right]$

$f(x) = 2 - \sin(\pi - x)$

$f(x) = 2 - \sin x,\ x \in \left(\frac{\pi}{2}, \pi\right]$

$f(x) = 2 - f(2\pi - x),\ \forall\ x \in \left(\pi, \frac{\pi}{2}\right]$

$f(x) = 2 + \sin x,\ \forall\ x \in \left(\pi, \frac{3\pi}{2}\right]$

$f(x) = f(2\pi - x),\ \forall\ x \in \left(\frac{3\pi}{2}, 2\pi\right]$

$f(x) = -\sin x,\ \forall\ x \in \left(\frac{3\pi}{2}, 2\pi\right]$

Clearly, from figure required area = 2π (स्पष्टतया, चित्र से अभीष्ट क्षेत्रफल = 2π)

27. The area bounded by the curves $y = x e^x$, $y = x e^{-x}$ and the line x = 1

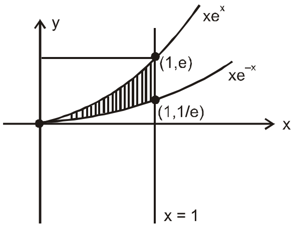
वक्रों $y = x e^x$, $y = x e^{-x}$ तथा रेखा x = 1 से परिबद्ध क्षेत्रफल है –

(A*) $\frac{2}{e}$ (B) $1 - \frac{2}{e}$ (C) $\frac{1}{e}$ (D) $1 - \frac{1}{e}$

Sol. $y = xe^x,\ y = xe^{-x}$

$xe^x = xe^{-x} \Rightarrow x(e^x - e^{-x}) = 0$

x = 0, Intersection point (0,0)



Figure

$$\text{Area} = \int_0^1 (xe^x - xe^{-x})\ \ dx$$

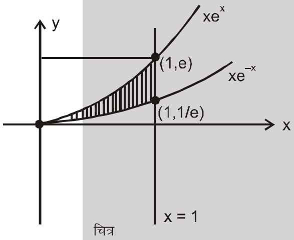
$$= x\ \ (e^x + e^{-x})_0^1 - \int_0^1 1.(e^x + e^{-x})\ \ dx$$

$$= \left(e + \frac{1}{e} - 0\right) - \left[e^x - e^{-x}\right]_0^1 = \frac{2}{e}$$

Hindi $y = xe^x$, $y = xe^{-x}$

$xe^x = xe^{-x} \Rightarrow x(e^x - e^{-x}) = 0$

x = 0, प्रतिच्छेद बिन्दु (0,0)



चित्र

$$\text{क्षेत्रफल} = \int_0^1 (xe^x - xe^{-x})\ \ dx$$

$$= x\ \ (e^x + e^{-x})_0^1 - \int_0^1 1.(e^x + e^{-x})\ \ dx$$

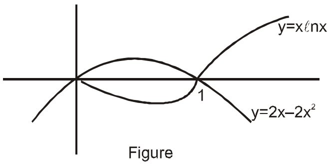
$$= \left(e + \frac{1}{e} - 0\right) - \left[e^x - e^{-x}\right]_0^1 = \frac{2}{e}$$

28. Obtain the area enclosed by region bounded by the curves $y = x\ \ell n\ x$ and $y = 2x - 2x^2$.

वक्र $y = x\ \ell n\ x$ और $y = 2x - 2x^2$ से परिबद्ध क्षेत्रफल है–

(A) 7/6 (B) 7/24 (C) 12/7 (D*) 7/12

Sol. $$\text{Area} = \int_0^1 (2x - 2x^2 - x\ \ \ell n\ \ x)\ \ dx$$

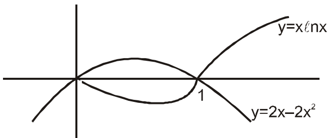


Figure

$$= x^2 - \frac{2}{3}x^3 - \frac{x^2}{2}\ \ \ell n\ x + \frac{x^2}{4}\Bigg|_0^1 = \frac{7}{12}$$

Hindi $$\text{क्षेत्रफल} = \int_0^1 (2x - 2x^2 - x\ \ \ell n\ \ x)\ \ dx$$

$$= x^2 - \frac{2}{3}x^3 - \frac{x^2}{2}\ \ell n\, x + \frac{x^2}{4}\Big|_0^1 = \frac{7}{12}$$



चित्र

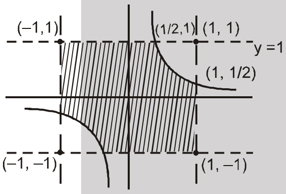
29. The area of the region on plane bounded by max (|x|, |y|) ≤ 1 and $xy \le \frac{1}{2}$ is

अधिकतम (|x|, |y|) ≤ 1 तथा $xy \le \frac{1}{2}$ से परिबद्ध क्षेत्र का क्षेत्रफल है –

(A) 1/2 + ℓn 2 (B*) 3 + ℓn 2 (C) 31/4 (D) 1 + 2 ℓn 2

Sol. $A_1 = 2\int_{1/2}^{1}\left(1 - \frac{1}{2x}\right)dx$

or $A_1 = 2\left(x - \frac{1}{2}\ell nx\right)_{1/2}^{1}$



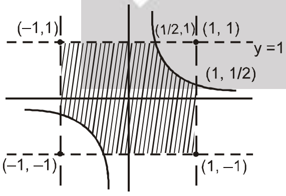
Figure

$\Rightarrow$ $A_1 = 1 - \ell n\, 2$

$\therefore$ area enclosed = 4 – (1 – ℓn 2) = 3 + ℓn 2

Hindi $A_1 = 2\int_{1/2}^{1}\left(1 - \frac{1}{2x}\right)dx$

या $A_1 = 2\left(x - \frac{1}{2}\ell nx\right)_{1/2}^{1}$



चित्र

$\Rightarrow$ $A_1 = 1 - \ell n\, 2$

$\therefore$ परिबद्ध क्षेत्रफल = 4 – (1 – ℓn 2) = 3 + ℓn 2

30. Consider the following statements : **[16JM120565]**

निम्नलिखित कथनों पर विचार कीजिए :

S_1 : The value of $\int_0^{2\pi}\cos^{-1}(\cos x)\ dx$ is π^2

S_1 : $\int_0^{2\pi} \cos^{-1}(\cos x)\ \ dx$ का मान π^2 है।

S_2 : Area enclosed by the curve $|x - 2| + |y + 1| = 1$ is equal to 3 sq. unit

S_2 : वक्र $|x - 2| + |y + 1| = 1$ से परिबद्ध क्षेत्रफल 3 वर्ग इकाई है।

S_3 : If $\frac{d}{dx}$ f(x) = g(x) for $a \le x \le b$, then $\int_a^b f(x)g(x)dx$ equals to $\frac{[f(b)]^2 - [f(a)]^2}{2}$.

S_3 : $a \le x \le b$ के लिए यदि $\frac{d}{dx}$ f(x) = g(x) हो, तो $\int_a^b f(x)g(x)dx$ का मान $\frac{[f(b)]^2 - [f(a)]^2}{2}$ है।

S_4 : Area of the region $R \equiv \{(x, y) ; x^2 \le y \le x\}$ is $\frac{1}{6}$

S_4 : क्षेत्र $R \equiv \{(x, y) ; x^2 \le y \le x\}$ का क्षेत्रफल $\frac{1}{6}$ है।

State, in order, whether S_1, S_2, S_3, S_4 are true or false

S_1, S_2, S_3, S_4 के सत्य (T) या असत्य (F) होने का सही क्रम है –

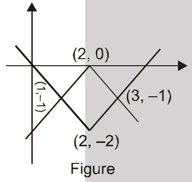
(A*) TFTT (B) TTTT (C) FFFF (D) TFTF

Sol. **S_1 :** $I = \int_0^{\pi} x dx + \int_{\pi}^{2\pi} (2\pi - x)\ \ dx\ = \frac{\pi^2}{2} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi} = \frac{\pi^2}{2} + \frac{\pi^2}{2} = \pi^2$

S_2 : $|y + 1| = 1 - |x - 2|$

$y = -1 \pm (1 - |x - 2|)$

$y = -|x - 2|\ (y \ge -1)$



Figure

$y = -2 + |x - 2|\ (y \le -1)$

Area $= \frac{1}{2}\begin{pmatrix} 2 & 3 & 2 & 1 & 2 \\ -2 & -1 & 0 & -1 & -2 \end{pmatrix}$ (formula from coordinate geometry)

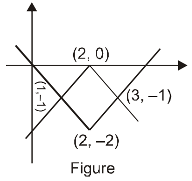
$= \frac{1}{2}\ (4) = 2$

Hindi **S_1 :** $I = \int_0^{\pi} x dx + \int_{\pi}^{2\pi} (2\pi - x)\ \ dx\ = \frac{\pi^2}{2} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi} = \frac{\pi^2}{2} + \frac{\pi^2}{2} = \pi^2$

S_2 : $|y + 1| = 1 - |x - 2|$

$y = -1 \pm (1 - |x - 2|)$

$y = -|x - 2|\ (y \ge -1)$



Figure

$y = -2 + |x - 2|\ (y \le -1)$

क्षेत्रफल $= \frac{1}{2}\begin{pmatrix} 2 & 3 & 2 & 1 & 2 \\ -2 & -1 & 0 & -1 & -2 \end{pmatrix}$ (निर्देशांक ज्यामिति के सूत्र से)

$= \frac{1}{2}\ (4) = 2$

Hindi माना $I_1 = \int_a^b f(x)\ \ g(x)dx$ [दिया है] $\frac{d}{dx}f(x) = g(x)$

$I_1 = \left[f(x)\int g(x)\ \ dx\right]_a^b - \int_a^b \left(\frac{d}{dx}f(x)\int\ g(x)\ \ dx\right)\ dx$

$= \left[f^2(x)\right]_a^b - \int_a^b f(x)\ \ g(x)\ \ dx \Rightarrow\ \ 2I_1 = [f(b)]^2 - [f(a)]^2 \quad \Rightarrow\ I_1 = \frac{[f(b)]^2 - [f(a)]^2}{2}$

S_4 : Area (क्षेत्रफल) = $\int_0^1 (x - x^2)dx\ = \frac{1}{6}$

$\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{6}$

PART-II: NUMERICAL VALUE QUESTIONS

भाग-II : संख्यात्मक प्रश्न (NUMERICAL VALUE QUESTIONS)

INSTRUCTION :

- The answer to each question is **NUMERICAL VALUE** with two digit integer and decimal upto two digit.
- If the numerical value has more than two decimal places **truncate/round-off** the value to **TWO** decimal placed.

निर्देश :

- इस खण्ड में प्रत्येक प्रश्न का उत्तर **संख्यात्मक मान** के रूप में है जिसमें दो पूर्णांक अंक तथा दो अंक दशमलव के बाद में है।
- यदि संख्यात्मक मान में दो से अधिक दशमलव स्थान है, तो संख्यात्मक मान को दशमलव के **दो** स्थानों तक **ट्रंकेट/राउंड ऑफ (truncate/round-off)** करें।

1. The value of integral $\int_2^4 \frac{3x^2+1}{(x^2-1)^3}dx$ is

समाकलन $\int_2^4 \frac{3x^2+1}{(x^2-1)^3}dx$ का मान है –

Ans. **00.20**

Sol. Note that दिया गया है $(x + 1)^3 - (x - 1)^3 = 2(3x^2 + 1)$

Hence अतः $I = \frac{1}{2}\left[\int_2^4 \frac{(x+1)^3 - (x-1)^3}{(x+1)^3(x-1)^3}dx\right] = \frac{1}{2}\left[\int_2^4 \frac{dx}{(x-1)^3} - \int_2^4 \frac{dx}{(x+1)^3}\right]$

$= \frac{1}{2}\left[\frac{1}{2(x+1)^2} - \frac{1}{2(x-1)^2}\right]_2^4 = \left[\left(\frac{1}{25} - \frac{1}{9}\right) - \left(\frac{1}{9} - 1\right)\right] = \frac{46}{225}$

2. Let U = $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \max\ (\sqrt{3}\sin x, \cos x)\ dx$ and V = $\int_{-3}^{5} x^2\ \text{sgn}\ (x-1)dx$. If V = λU, then find the value of λ.

[**Note :** sgn k denotes the signum function of k.] **[16JM120566]**

माना U = $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \max\ (\sqrt{3}\sin x, \cos x)\ dx$ और V = $\int_{-3}^{5} x^2\ \text{sgn}\ (x-1)dx$ यदि V = λU तब λ का मान ज्ञात कीजिए।

[**Note :** जहाँ sgn k, k के सिग्नम फलन को व्यक्त करता है।]

Ans. **21.33**

Sol. U = $\int_{\pi/6}^{\pi/2} \sqrt{3}\sin x\, dx = -\sqrt{3}(\cos x)\Big|_{\pi/6}^{\pi/2} = \frac{3}{2}$

V = $\int_{-3}^{1} (-x^2)dx + \int_{1}^{5} x^2 dx = \frac{1}{3}\left[-(1+27)+(125-1)\right] = \frac{1}{3}[-28+124] = 32$

∴ V = λU ⇒ $\lambda = \frac{64}{3}$

3. ✍ Let f(x) be a function satisfying $f(x) = f\left(\frac{100}{x}\right)\ \forall x > 0$. If $\int_{1}^{10} \frac{f(x)}{x}dx = 5$ then find the value of $\int_{1}^{100} \frac{f(x)}{x}dx$

माना कि फलन f(x), $f(x) = f\left(\frac{100}{x}\right)\ \forall x > 0.$ को सन्तुष्ट करता है यदि $\int_{1}^{10} \frac{f(x)}{x}dx = 5$ हो तो $\int_{1}^{100} \frac{f(x)}{x}dx$ का मान ज्ञात कीजिए।

Ans. **10.00**

Sol. $\int_{1}^{100} \frac{f(x)}{x}dx = \int_{1}^{10} \frac{f(x)}{x}dx + \int_{10}^{100} \frac{f(x)}{x}dx = 5 + \int_{10}^{1} \frac{f\left(\frac{100}{t}\right)}{100}.t\left(\frac{-100dt}{t^2}\right)$ (substituting रखने पर $x = \frac{100}{t}$)

$= 5 + \int_{t}^{10} f\left(\frac{100}{t}\right).\frac{dt}{t} = 5 + \int_{1}^{10} \frac{f(t)}{t}dt = 10$

4. Evaluate **[16JM120567]**

$$\frac{2005\int_{0}^{1002} \frac{dx}{\sqrt{1002^2 - x^2} + \sqrt{1003^2 - x^2}} + \int_{1002}^{1003} \sqrt{1003^2 - x^2}\, dx}{\int_{0}^{1} \sqrt{1 - x^2}\, dx}$$ = k, then find the sum of squares of digits of natural number k.

सरल कीजिए

$$\frac{2005\int\limits_0^{1002}\frac{dx}{\sqrt{1002^2-x^2}+\sqrt{1003^2-x^2}}+\int\limits_{1002}^{1003}\sqrt{1003^2-x^2}\,dx}{\int\limits_0^1\sqrt{1-x^2}\,dx} = k,$$ तो प्राकृत संख्या k के अंको के वर्गो का योगफल ज्ञात कीजिए।

Ans. **29.00**

Sol. $$\frac{2005\int\limits_0^{1002}\frac{\sqrt{1003^2-x^2}-\sqrt{1002^2-x^2}}{2005}+\int\limits_{1002}^{1003}\sqrt{1003^2-x^2}\,dx}{\int\limits_0^1\sqrt{1-x^2}\,dx}$$

$$=\frac{\int\limits_0^{1003}\sqrt{1003^2-x^2}\,dx-\int\limits_0^{1002}\sqrt{1002^2-x^2}\,dx}{\int\limits_0^1\sqrt{1-x^2}\,dx}$$

Hence अतः $2^2 + 5^2 = 29$ है

5.✍ The value of integral $\int\limits_0^{\pi/2}\sqrt{\sin2\theta}.\sin\theta\,d\theta$ is

समाकलन $\int\limits_0^{\pi/2}\sqrt{\sin2\theta}.\sin\theta\,d\theta$ का मान है –

Ans. **00.78**

Sol. $I = \int\limits_0^{\pi/2}\sqrt{\sin2\theta}\,\cos\theta\,d\theta$; $I = \int\limits_0^{\pi/2}\sqrt{\sin2\theta}\,\sin\theta\,d\theta$ (By property गुणधर्म से)

Let माना $\sin\theta - \cos\theta = t$; $2I = \int\limits_{-1}^{1}\sqrt{1-t^2} = 2\int\limits_0^1\sqrt{1-t^2}\,dt$;

$I = \int\limits_0^1\sqrt{1-t^2}\,dt = \frac{\pi}{4}$

6.✍ Let $I_1 = \int\limits_0^{\pi/4}(1+\tan x)^2dx$, $I_2 = \int\limits_0^1\frac{dx}{(1+x)^2(1+x^2)}$ **[16JM120568]**

then find the value of $\frac{I_1}{I_2}$

माना $I_1 = \int\limits_0^{\pi/4}(1+\tan x)^2dx$, $I_2 = \int\limits_0^1\frac{dx}{(1+x)^2(1+x^2)}$ तब $\frac{I_1}{I_2}$ का मान ज्ञात कीजिए।

Ans. **04.00**

Sol. Substitute $x = \tan\theta$ in I_2. I_2 मे $x = \tan\theta$ रखने पर

$$I_2 = \int_0^{\pi/4} \frac{d\theta}{(1+\tan\theta)^2} = \int_0^{\pi/4} \frac{d\theta}{\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)^2} \qquad \left[\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$$

$$= \int_0^{\pi/4} \frac{(1+\tan\theta)^2}{4} d\theta \qquad \Rightarrow \qquad I_2 = \frac{I_1}{4}$$

7. Find the value of $\left(\int_0^1 e^{t^2+t}(2t^2+t+1)dt\right)$

$\left(\int_0^1 e^{t^2+t}(2t^2+t+1)dt\right)$ का मान ज्ञात कीजिए।

Ans. **07.38 or 07.39**

Sol. $I = \int_0^1 \underset{f(t)}{e^{t^2+t}} + \underbrace{t.e^{t^2+t}(2t+1)}_{f'(t)} dt \; = \; t.e^{t^2+t}\Big|_0^1 = e^2$

8. If $\int_0^1 \frac{x}{x+1+e^x}dx$ is equal to ℓnk , then find the value of k. **[16JM120569]**

यदि $\int_0^1 \frac{x}{x+1+e^x}dx$ का मान ℓnk बराबर है तब k का मान ज्ञात कीजिए।

Ans. **01.15**

Sol. Let माना $I = \int_0^1 \frac{xe^{-x}}{(x+1)e^{-x}+1}dx$ substituting $1 + (x + 1)e^{-x} = t$ रखने पर

$\Rightarrow \quad -xe^{-x}\, dx = dt$

$$I = -\int_2^{1+\frac{2}{e}} \frac{dt}{t} = \ell n\left(\frac{2e}{e+2}\right)$$

$$k = \left(\frac{2e}{e+2}\right) = a = 01.15$$

9. If f, g, h be continuous functions on [0, a] such that f (a – x) = f (x), g (a – x) = – g (x) and 3 h (x) – 4 h (a – x) = 5, then find the value of $\int_0^a$ f (x) g (x) h (x) dx

Ans. 00.00

यदि f, g, h अन्तराल [0, a] में सतत् फलन इस प्रकार है कि f (a – x) = f (x), g (a – x) = – g (x) और 3 h (x) – 4 h (a – x) = 5, तब $\int_0^a$ f (x) g (x) h (x) dx का मान ज्ञात करें।

Sol. $I = \int_0^a f(x)\ g(x)\ h(x)\ dx$

$$I = \int_0^a f(a-x)\ g(a-x)\ h(a-x)\ dx = \int_0^a f(x)\ .\ (-g(x))\ \frac{3h(x)-5}{4}\ dx$$

$$I = -\ \frac{3}{4}\int_0^a f(x)\ g(x)\ h(x) + \frac{5}{4}\int_0^a f(x)\ g(x)\ dx$$

$$I = -\ \frac{3}{4}\ I + 0\ \frac{7}{4}\ I = 0$$

$\frac{7}{4}\ I = 0 \qquad \Rightarrow \qquad I = 0 \qquad\qquad \because \qquad f(x)\ g(x)\ dx = 0$

$I_1 = \int_0^a f(x)\ g(x)\ dx \qquad$ ......(i)

$I_1 = \int_0^a f(a-x)\ g(a-x)\ dx$

$I_1 = \int_0^a f(x)\ (-g(x))\ dx \qquad$ ......(ii)

(i) + (ii) $2I_1 = 0 \Rightarrow \qquad I_1 = 0$

10. If $f(x) = \frac{\sin x}{x}$ ∀ x ∈ (0, π], If $k\int_0^{\pi/2} f(x)\ f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi} f(x)\ dx$ then find the value of k. **[16JM120570]**

यदि $f(x) = \frac{\sin x}{x}$ ∀ x ∈ (0, π] हो, तथा $k\int_0^{\pi/2} f(x)\ f\left(\frac{\pi}{2}-x\right)dx = \int_0^{\pi} f(x)\ dx$ है, तब k का मान ज्ञात करो।

Ans. **01.57**

Sol. $I = \int_0^{\pi} f(x)\ dx = \int_0^{\pi}\frac{\sin x}{x}\ dx$ ... (i)

$I = \int_0^{\pi} f(\pi-x)\ dx = \int_0^{\pi}\frac{\sin(\pi-x)}{\pi-x}\ dx = \int_0^{\pi}\frac{\sin x}{\pi-x}\ dx$ ... (ii)

(i) + (ii)

$\Rightarrow \quad 2I = \int_0^{\pi}\left\{\frac{\sin x}{x}+\frac{\sin x}{\pi-x}\right\}dx \Rightarrow I = \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{x(\pi-x)}dx$ ... (iii)

Now (अब) $\frac{\pi}{2}\int_0^{\frac{\pi}{2}} f(x)\ f\left(\frac{\pi}{2}-x\right)\ dx = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}\ \frac{\sin\left(\frac{\pi}{2}-x\right)}{\frac{\pi}{2}-x}\ dx$

$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}\cdot\frac{\cos x}{\frac{\pi}{2}-x}\ dx = \cdot\frac{\pi}{4}\int_0^{\pi/2}\frac{\sin\ \ 2x}{x\left(\frac{\pi}{2}-x\right)}\ dx$

$= \frac{\pi}{8}\int_0^{\pi}\frac{\sin t}{\frac{t}{2}\left(\frac{\pi}{2}-\frac{t}{2}\right)}dt$, where (जहाँ) t = 2x ... (iv)

(iii) + (iv)

$\Rightarrow \frac{\pi}{2}\int_0^{\pi} f(x)\ f\left(\frac{\pi}{2}-x\right)\ dx = \int_0^{\pi} f(x)\ dx$

11. Evaluate: $\int_0^{\pi}\frac{a^2\sin^2 x+b^2\cos^2 x}{a^4\sin^2 x+b^4\cos^2 x}dx$, where $a^2 + b^2 = \frac{3\pi}{4}$, $a^2 \neq b^2$ and $ab \neq 0$.

सरल कीजिए : $\int_0^{\pi}\frac{a^2\sin^2 x+b^2\cos^2 x}{a^4\sin^2 x+b^4\cos^2 x}dx$, जहाँ $a^2 + b^2 = \frac{3\pi}{4}$, $a^2 \neq b^2$ और $ab \neq 0$.

Ans. **02.66 or 02.67**

Sol. Let माना $I = \frac{2}{a^2+b^2}\int_0^{\pi/2}\frac{(a^2+b^2)(a^2\sin^2 x+b^2\cos^2 x)}{a^4\sin^2 x+b^4\cos^2 x}dx = \frac{2}{3\pi}.4\int_0^{\pi/2}\left(\frac{a^4\cos^2 x+b^4\cos^2 x+a^2b^2}{a^4\sin^2 x+b^4\cos^2 x}\right)dx$

$= \frac{8}{3\pi}\int_0^{\pi/2}dx + \frac{8a^2b^2}{3\pi}\int_0^{\pi/2}\frac{\sec^2 x}{a^4\sin^2 x+b^4\cos^2 \pi}dx$

$= \frac{4}{3}+\frac{8}{3\pi}\frac{b^2}{a^2}\int_0^{\pi/2}\frac{\sec^2 x\,dx}{\tan^2 x+(b^2/a^2)^2} = \frac{4}{3}+\frac{8b^2}{3\pi a^2}.\frac{a^2}{b^2}.\tan^{-1}\left(\frac{a^2\tan x}{b^2}\right)_0^{\pi/2} = \frac{4}{3}+\frac{4}{3}=\frac{8}{3}$

12. $\int_{0}^{2\pi} |\sqrt{15}\sin x + \cos x|\,dx$ **[16JM120571]**

Ans. **16.00**

Sol. $I = \int_{0}^{2\pi} |\sin(x+\alpha)|\,dx$, where जहाँ $\tan\alpha = \dfrac{1}{\sqrt{15}}$

$= 4\int_{\alpha}^{2\pi+\alpha} |\sin|\,dt$ (substituting रखने पर $x + \alpha = t$)

$= \left(\int_{a}^{a+T} f(x)dx = \int_{0}^{T} f(x)dx \text{ if } f(x+T) = f(x)\right)$

$= 16$

13. Let a be a real number in the interval [0, 314] such that $\int_{-\pi+a}^{3\pi+a} |x-a-\pi|\sin\left(\dfrac{x}{2}\right)dx = -16$, then determine number of such values of a.

माना a वास्तविक संख्या अंतराल [0, 314] मे इस प्रकार है कि $\int_{-\pi+a}^{3\pi+a} |x-a-\pi|\sin\left(\dfrac{x}{2}\right)dx = -16$, तब इस प्रकार के a के मानों की संख्या ज्ञात कीजिए।

Ans. **25.00**

Sol. Let माना $I = \int_{-\pi+a}^{3\pi+a} |x-a-\pi|\sin\left(\dfrac{x}{2}\right)dx$ Let माना $x - a - \pi = t$

$\Rightarrow I = \int_{-2\pi}^{2\pi} |t|\sin\left(\dfrac{t+a}{2}+\dfrac{\pi}{2}\right)dt = \int_{-2\pi}^{2\pi} |t|\cos\left(\dfrac{a+t}{2}\right)dt = \int_{-2\pi}^{2\pi} t\left\{\cos\left(\dfrac{a+t}{2}\right)+\cos\left(\dfrac{a-t}{2}\right)\right\}dt$

$= \int_{-2\pi}^{2\pi} t\left\{2\cos\dfrac{a}{2}\cos\dfrac{t}{2}\right\}dt = 2\cos\left(\dfrac{a}{2}\right)\int_{0}^{2\pi}\cos\left(\dfrac{t}{2}\right)dt$ Let $= \dfrac{t}{2}\ y$

$8\cos\left(\dfrac{a}{2}\right)\int_{0}^{\pi} y\cos y\,dy = 8\cos\left(\dfrac{a}{2}\right)\{y\sin y + \cos y\}_{0}^{\pi} = -16\cos\left(\dfrac{a}{2}\right)$

Now अब $I = -16 \Rightarrow \cos\left(\dfrac{a}{2}\right) = 1 \Rightarrow a = 4k\pi,\ k \in I$

Hence number of value of 'a' is 25. अतः 'a' के मानों की संख्या 25 है।

14.✍ Value of $\sum_{n=1}^{\infty}\left(\dfrac{1}{4n-3}-\dfrac{1}{4n-1}\right)$ is $\left(\text{Note that } \tan^{-1}x + c = \int \dfrac{1}{1+x^2}dx\right)$ **[16JM120572]**

$\sum_{n=1}^{\infty}\left(\dfrac{1}{4n-3}-\dfrac{1}{4n-1}\right)$ का मान होगा $\left(\text{दिया गया है } \tan^{-1}x + c = \int \dfrac{1}{1+x^2}dx\right)$

Ans. **00.78**

Sol. $\int_{0}^{1}\dfrac{dx}{1+x^2} = \int_{0}^{1}(1 - x^2 + x^4 - x^6 + x^8 - x^{10} + ...)dx$

$\Rightarrow \tan^{-1}x\Big|_{0}^{1} = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9}.......\Big|_{0}^{1} \Rightarrow \dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \dfrac{1}{9}.......$

15. If f (x) = x + $\int_0^1 t(x+t)\ f(t)\ dt$, then the value of the definite integral $\int_0^1 f(x)\ dx$ is

यदि f (x) = x + $\int_0^1 t(x+t)\ f(t)\ dt$ तब $\int_0^1 f(x)\ dx$ का मान होगा –

Ans. **01.82 or 01.83**

Sol. f(x) = x + $\int_0^1 t(x+t)\, f(t)dt$; f(x) = x + x$\int_0^1 t f(t)dt + \int_0^1 t^2 f(t)dt$

Let माना $\int_0^1 t f(t)dt = a$ & तथा $\int_0^1 t^2 f(t)dt = b$ Hence अतः f(x) = (a + 1)x + b

so इसलिए a = $\int_0^1 t\{(a+1)t+b\}dt = \frac{a+1}{3}+\frac{b}{2}$ ⇒ 4a – 3b = 2 ....(1)

& तथा b = $\int_0^1 t^2\{(a+1)t+b\}\,dt = \frac{a+1}{4}+\frac{b}{3}$ ⇒ 8b – 3a = 3 ....(2)

from (1) & और (2) से a = $\frac{25}{23}$ & तथा b = $\frac{18}{23}$

so $\int_0^1 f(t)dt = \frac{6}{23}\int_0^1 (8t+3)dt = \frac{6}{23}.7 = \frac{42}{23}$

16. If f(x) = (ax + b) e^x satisfies the equation : $f(x) = \int_0^x e^{x-y} f'(y)dy - (x^2 - x + 1)e^x$, find $(a^2 + b^2)$

यदि f(x) = (ax + b) e^x समीकरण : $f(x) = \int_0^x e^{x-y} f'(y)dy - (x^2 - x + 1)e^x$ को सन्तुष्ट करती है तब $(a^2 + b^2)$ का मान ज्ञात कीजिए।

Ans. **05.00**

Sol. $f(x).e^{-x} = \int_0^x e^{-y}.e^{y}(ay + b + a)dy - (x^2 + x + 1)$

$f'(x).e^{-x} - f(x).e^{-x} = ax + (a + b) - (2x + 1)$

$f'(x) - f(x) = e^x\{(a - 2)x + (a + b - 1)\}$

⇒ $\{(ax + b + a) - (ax + b)\}e^x = e^x\{(a - 2)x + (a + b - 1)\}$

⇒ a – 2 = 0 & a = a + b – 1⇒ a = 2 & b = 1

⇒ $a^2 + b^2 = 5$

17. If the minimum of the following function f(x) defined at 0 < x < $\frac{\pi}{2}$.

f(x) = $\int_0^x \frac{d\theta}{\cos\theta} + \int_x^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}$ is equal to $\ell n(k)$ then value of k is

यदि 0 < x < $\frac{\pi}{2}$ मे परिभाषित निम्न फलन f(x) = $\int_0^x \frac{d\theta}{\cos\theta} + \int_x^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}$ का न्यूनतम मान $\ell n(k)$ हो तो k का मान होगा –

Ans. **05.82 or 05.83**

Sol. $f(x) = \int_0^x \frac{d\theta}{\cos\theta} + \int_x^{\pi/2} \frac{d\theta}{\sin\theta}$ $\Rightarrow$ $f'(x) = \frac{1}{\cos x} - \frac{1}{\sin x} = \frac{\sin x - \cos x}{\sin x - \cos x}$

$f'(x) = 0$ $\Rightarrow$ $x = \frac{\pi}{4}$ $\Rightarrow$ $f'(x)$ changes sign from (–) to (+) hence it is a point of minima.

Now अब $f = \left(\frac{\pi}{4}\right) = \int_0^{\pi/4} \sec\theta d\theta + \int_{\pi/4}^{\pi/2} \text{cosec } d\theta = 2\int_0^{\pi/4} \sec\theta d\theta = 2\ln(\sec\theta + \tan\theta)\Big|_0^{\pi/4} = 2\ell n(\sqrt{2} + 1)$

$= \ell n(3 + \sqrt{8})$

Hence अत: a + b = 11

18. If $f(\pi) = \pi$ and $\int_0^{\pi} (f(x) + f''(x)) \sin x\, dx = 7\pi$, then find the value of f(0) **[16JM120574]**

(it is given that f(x) is continuous in $[0, \pi]$)

यदि $f(\pi) = \pi$ और $\int_0^{\pi} (f(x) + f''(x)) \sin x\, dx = 7\pi$, तब f(0) का मान ज्ञात कीजिए।

(यह दिया गया है कि f(x), $[0, \pi]$ में सतत है।)

Ans. **18.84 or 18.85**

Sol. $7\pi = \int_0^{\pi} f(x) . \sin x\ dx + \int_0^{\pi} \sin x . f''(x)\ dx$

$7\pi = -(f(x) . \cos x)^{\pi}_0 + \int_0^{\pi} f'(x) \cos x dx + (\sin x . f'(x))_0^{\pi} - \int_0^{\pi} \cos x\ f'(x)\ dx$

$7\pi - \pi = f(0) \Rightarrow f(0) = 6\pi$

19. ✍ Let $f(x) = 2x^3 - 15x^2 + 24x$ and $g(x) = \int_0^x f(t)\ dt + \int_0^{5-x} f(t)\ dt$ $(0 < x < 5)$. If g(x) is increasing in the interval (0,p] then value of p is

माना $f(x) = 2x^3 - 15x^2 + 24x$ तथा $g(x) = \int_0^x f(t)\ dt + \int_0^{5-x} f(t)\ dt$ $(0 < x < 5)$ है। यदि g(x) अन्तराल (0,p] में वर्धमान फलन हो तो p का मान ज्ञात कीजिए –

Ans. 02.50

Sol. $f'(x) = 6(x^2 - 5x + 4)$

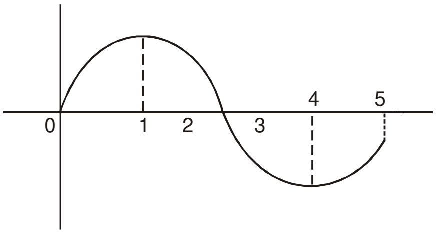
$g'(x) = f(x) - f(5 - x)$

for function g(x) to be increasing

$g'(x) > 0$

Now graph of f(x) will be as shown

If $0 < x < \frac{5}{2}$



$f(x) > f(5 - x)$

So g(x) is increasing in $\left(0, \frac{5}{2}\right]$. Hence 2 integers 1 and 2.

Hindi (ii) $f'(x) = 6(x^2 - 5x + 4)$

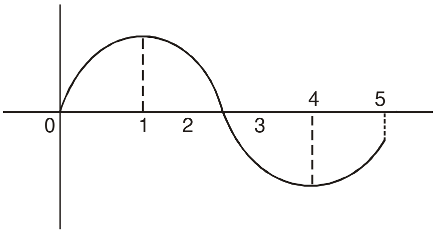
$g'(x) = f(x) - f(5 - x)$

g(x) के वर्द्धमान होने के लिए

$g'(x) > 0$

अतः f(x) का आलेख है

यदि $0 < x < \frac{5}{2}$; $f(x) > f(5 - x)$



अतः g(x) वर्द्धमान है $\left(0, \frac{5}{2}\right]$ अतः दो पूर्णांक है 1 एवं 2.

20. Let $f(x) = \begin{cases} 1-x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \\ (2-x)^2 & \text{if } 2 < x \le 3 \end{cases}$ and function $F(x) = \int_0^x f(t)\,dt$. If number of points of discontinuity in [0, 3] and non-differentiablity in (0, 3) of F(x) are α and β respectively, then (α – β) is equal to.

Ans. **00.00**

माना कि $f(x) = \begin{cases} 1-x & \text{यदि } 0 \le x \le 1 \\ 0 & \text{यदि } 1 < x \le 2 \\ (2-x)^2 & \text{यदि } 2 < x \le 3 \end{cases}$, फलन $F(x) = \int_0^x f(t)\,dt$ है। यदि F(x) के अंतराल [0, 3] में असतता के बिन्दुओं की संख्या α है अन्तराल (0, 3) में अवकलनीय नही होने के बिन्दुओं की संख्या β है। तब (α – β) का मान बराबर है।

Ans. $F(x) = \begin{cases} x - \frac{x^2}{2} & \text{if } 0 \le x \le 1 \\ \frac{1}{2} & \text{if } 1 < x \le 2 \\ \frac{(x-2)^3}{3} + \frac{1}{2} & \text{if } 2 < x \le 3 \end{cases}$

$F(x) = \begin{cases} x - \frac{x^2}{2} & \text{यदि } 0 \le x \le 1 \\ \frac{1}{2} & \text{यदि } 1 < x \le 2 \\ \frac{(x-2)^3}{3} + \frac{1}{2} & \text{यदि } 2 < x \le 3 \end{cases}$

Sol. $F(x) = \begin{bmatrix} \int_0^x (1-t)\,dt & ; & 0 \le x \le 1 \\ \int_0^1 (1-t)\,dt + \int_1^x 0\,.\,dx & ; & 1 < x \le 2 \\ \int_0^1 (1-t)\,dt + \int_1^2 0\,.\,dx + \int_2^x (2-t)^2\,dt & ; & 2 < x \le 3 \end{bmatrix}$

; $F(x) = \begin{bmatrix} x - \frac{x^2}{2} & ; & 0 \le x \le 1 \\ \frac{1}{2} & ; & 1 < x \le 2 \\ \frac{(x-2)^3}{3} + \frac{1}{2} & ; & 2 < x \le 3 \end{bmatrix}$

21. Find the value of m (m > 0) for which the area bounded by the line y = mx + 2 and $x = 2y - y^2$ is $\frac{256}{3}$ square units. **[16JM120575]**

m (m > 0) वे वह मान ज्ञात कीजिए जिनके लिए रेखा y = mx + 2 तथा वक्र $x = 2y - y^2$ से परिबद्ध क्षेत्र का क्षेत्रफल $\frac{256}{3}$ वर्ग इकाई है।

Ans. **00.16 or 00.17**

Sol. $y = mx + 2 \Rightarrow x = \left(\frac{y-2}{m}\right)$ ....(1)

$x = 2y - y^2$ ....(2)

$(y-1)^2 = -(x-1)$ vertex (1, 1)

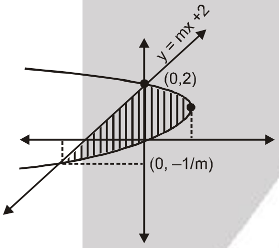
From (1) and (2) $\frac{y-2}{m} = 2y - y^2$

$\Rightarrow my^2 + (1-2m)y - 2 = 0$ $\alpha\beta = -\frac{2}{m}$

$\alpha = 2, \quad \beta = -\frac{1}{m}$

$$\text{Area} = \int_{-1/m}^{2}\left[\left(2y - y^2\right) - \frac{(y-2)}{m}\right] dy$$

$$\frac{256}{3} = \left[\frac{2y^2}{2} - \frac{y^3}{3} - \frac{1}{m}\frac{y^2}{2} + \frac{2y}{m}\right]_{-1/m}^{2} \Rightarrow \frac{256}{3} = \left(\frac{4}{3} + \frac{2}{m} + \frac{1}{6m^3} + \frac{1}{m^2}\right)$$



Figure

$m = \frac{1}{6}$ satisfy the equation

Hindi $y = mx + 2 \Rightarrow x = \left(\frac{y-2}{m}\right)$ ....(1)

$x = 2y - y^2$ ....(2)

$(y-1)^2 = -(x-1)$ शीर्ष (1, 1)

(1) तथा (2) से $\frac{y-2}{m} = 2y - y^2$

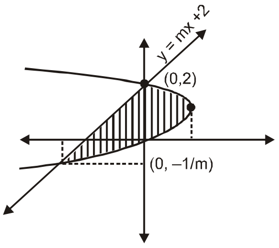
$\Rightarrow my^2 + (1-2m)y - 2 = 0$ $\alpha\beta = -\frac{2}{m}$

$\alpha = 2, \quad \beta = -\frac{1}{m}$

$$\text{क्षेत्रफल} = \int_{-1/m}^{2}\left[\left(2y - y^2\right) - \frac{(y-2)}{m}\right] dy$$

$$\frac{256}{3} = \left[\frac{2y^2}{2} - \frac{y^3}{3} - \frac{1}{m}\frac{y^2}{2} + \frac{2y}{m}\right]_{-1/m}^{2}$$

$$\Rightarrow \quad \frac{256}{3} = \left(\frac{4}{3}+\frac{2}{m}+\frac{1}{6m^3}+\frac{1}{m^2}\right)$$



Figure

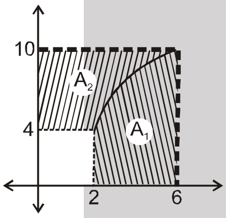
$m = \frac{1}{6}$ समीकरण को संतुष्ट करता है

22. Find area bounded by $y = f^{-1}(x)$, x = 10, x = 4 and x-axis given that area bounded by y = f(x), x = 2, x = 6 and x-axis is 30 sq. units, where f(2) = 4 and f(6) = 10. (given f(x) is an invertible function) **Ans.** 22 sq. units

$y = f^{-1}(x)$, x = 10, x = 4 और x-अक्ष से परिबद्ध क्षेत्रफल ज्ञात कीजिए। दिया गया है कि y = f(x), x = 2, x = 6 और x-अक्ष से परिबद्ध क्षेत्रफल 30 वर्ग इकाई जहाँ f(2) = 4 और f(6) = 10 दिया गया है कि f(x) प्रतिलोमीय फलन है।

Ans. **22.00**

Sol. Now as per the above information. अब ऊपर लिखी सूचना से



$\int_4^{10} f^{-1}(x)dx = A_2$, $\quad A_1 = \int_2^6 f(x)dx$

Now अब $A_2 + A_1 = 60 - 8 = 52 \quad \Rightarrow \quad A_2 = 22$ (as चूँकि $A_1 = 30$)

23. Consider a line $\ell : 2x - \sqrt{3}y = 0$ and a parameterized $C : x = \tan t,\ y = \frac{1}{\cos t}\left(0 \le t < \frac{\pi}{2}\right)$

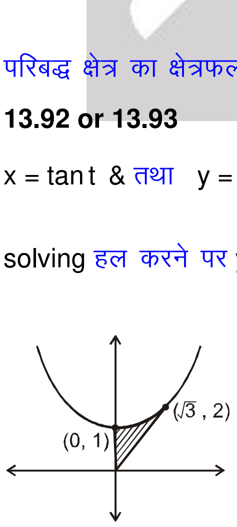
If the area of the part bounded by ℓ, C and the y-axis is equal to $\frac{1}{4}\ell n(k)$, where a, b, ∈ N, b, is not perfect square then find the value of (a + b) **[16JM120576]**

माना कि एक रेखा $\ell : 2x - \sqrt{3}y = 0$ तथा प्राचलिक $C : x = \tan t,\ y = \frac{1}{\cos t}\left(0 \le t < \frac{\pi}{2}\right)$ यदि ℓ, C और y-अक्ष से परिबद्ध क्षेत्र का क्षेत्रफल $\frac{1}{4}\ell n(k)$ है जहाँ a, b, ∈ N, b एक पूर्ण वर्ग नही है तब (a + b) का मान ज्ञात कीजिए।

Ans. **13.92 or 13.93**

Sol. $x = \tan t$ & तथा $y = \sec t \quad \left(t \in \left(0, \frac{\pi}{2}\right)\right) \quad \Rightarrow \quad y^2 - x^2 = 1 \quad (x > 0, y > 1)$

solving हल करने पर $y = \frac{2x}{\sqrt{3}}$ and और (tan t, sec t), we get t = π/3



Area क्षेत्रफल = $\int_0^1 \frac{\sqrt{3}\,y}{2}dt + \int_0^2\left(\frac{\sqrt{3}\,y}{2} - \sqrt{y^2-1}\right)dy = \frac{\sqrt{3}}{4}.y^2\Big|_0^2 - \left(\frac{y}{2}\sqrt{y^2-1} - \frac{1}{2}\ell n(y+\sqrt{y^2-1})\right)\Big|_1^2$

$= \sqrt{3} - \left\{\sqrt{3} - \frac{1}{2}\ell n(2+\sqrt{3})\right\} = \frac{1}{4}\ell n(7+\sqrt{48}) \Rightarrow a + b = 55$

PART - III : ONE OR MORE THAN ONE OPTIONS CORRECT TYPE

भाग - III : एक या एक से अधिक सही विकल्प प्रकार

1. $\int_0^\infty \frac{x}{(1+x)(1+x^2)}$ dx equals to :

(A*) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C*) is same as $\int_0^\infty \frac{dx}{(1+x)(1+x^2)}$

(D) cannot be evaluated

$\int_0^\infty \frac{x}{(1+x)(1+x^2)}$ dx का मान है–

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C) $\int_0^\infty \frac{dx}{(1+x)(1+x^2)}$ के समान है

(D) गणना नहीं की जा सकती

Sol. $\int_0^\infty \frac{x\ \ dx}{(1+x)(1+x^2)}$, x = tanθ ; $I = \int_0^{\pi/2} \frac{\tan\theta}{(1+\tan\theta).\sec^2\theta}.\sec^2\theta d\theta$

$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}d\theta$ ....(i) ; $I = \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta+\cos\theta}d\theta$ ....(ii)

(i) + (ii)

$2I = \int_0^{\pi/2} 1.d\theta = \frac{\pi}{2}$ $\quad$ $I = \pi/4$

$\int_0^\infty \frac{dx}{(1+x)(1+x^2)}$ $\quad$ x = tanθ ; $I = \int_0^{\pi/2} \frac{\sec^2\theta\ \ d\theta}{(1+\tan\theta).\sec^2\theta}$.

$I = \int_0^{\pi/2} \frac{\sin\theta}{\sin\theta+\cos\theta}d\theta = \frac{\pi}{4}$ (as above) (जैसा ऊपर दिया है)

2. The value of integral $\int_a^b \frac{|x|}{x}$ dx, a < b is : **[16JM120577]**

(A*) b – a if a > 0 $\quad$ (B*) a – b if b < 0 $\quad$ (C*) b + a if a < 0 < b $\quad$ (D*) | b | – | a |

समाकल $\int_a^b \frac{|x|}{x}$dx, a < b, का मान है–

(A) b – a यदि a > 0 $\quad$ (B) a – b यदि b < 0 $\quad$ (C) b + a यदि a < 0 < b $\quad$ (D) | b | – | a |

Sol. Case-I $0 < a < b$

|x| = x then $\int_a^b \frac{|x|}{x} dx = b - a$

Case- II a < 0 < b then $\int_a^b \frac{|x|}{x} dx = \int_a^0 \frac{|x|}{x} dx + \int_0^b \frac{|x|}{x} dx = a + b$

Case-III a < b < 0 then $\int_a^b \frac{|x|}{x} dx = \int_a^b (-1)\ dx = a - b \quad \therefore \int_a^b \frac{|x|}{x} dx \quad = |b| - |a|$

Hindi स्थिति-I 0 < a < b

|x| = x तब $\int_a^b \frac{|x|}{x} dx = b - a$

स्थिति- II a < 0 < b तब $\int_a^b \frac{|x|}{x} dx = \int_a^0 \frac{|x|}{x} dx + \int_0^b \frac{|x|}{x} dx = a + b$

स्थिति-III a < b < 0 तब $\int_a^b \frac{|x|}{x} dx = \int_a^b (-1)\ dx = a - b \quad \therefore \int_a^b \frac{|x|}{x} dx \quad = |b| - |a|$

3. If $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$; n ∈ N, then which of the following statements hold good?

यदि $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$; n ∈ N, तब निम्न में से कौन–कौन से कथन सत्य है–

(A*) $2n\ I_{n+1} = 2^{-n} + (2n - 1)\ I_n$ (B*) $I_2 = \frac{\pi}{8} + \frac{1}{4}$

(C) $I_2 = \frac{\pi}{8} - \frac{1}{4}$ (D) $I_3 = \frac{\pi}{16} - \frac{5}{48}$

Sol. $I_n = \int_0^1 \frac{dx}{(1+x^2)^n} = \int_0^1 (1+x^2)^{-n} dx = \left[\frac{x}{(1+x^2)^n}\right]_0^1 - \int_0^1 (-n)(1+x^2)^{-n-1} 2x^2\, dx$

$= \frac{1}{2^n} + 2\ \ n \int_0^1 \frac{x^2}{(1+x^2)^{n+1}} = \frac{1}{2^n} + 2\ \ n \int_0^1 \frac{1+x^2-1}{(1+x^2)^{n+1}} dx = \frac{1}{2^n} + 2\ \ n\ \ I_n - 2n\ \ I_{n+1}$

$\therefore \quad 2n\ I_{n+1} = 2^{-n} + (2n - 1)\ I_n \qquad \therefore \quad 2I_2 = \frac{1}{2} + I_1 = \frac{1}{2} + \left[\tan^{-1} x\right]_0^1 \Rightarrow \quad I_2 = \frac{1}{4} + \frac{\pi}{8}$

4. The value of integral $\int_0^\pi xf\ (\sin x)\, dx$ is :

(A*) $\frac{\pi}{2} \int_0^\pi f\ (\sin x)\ dx$ (B*) $\pi \int_0^{\pi/2} f\ (\sin x)\ dx$ (C) 0 (D) $2\pi \int_0^{\pi/4} f\ (\sin x)\, dx$

समाकल $\int_0^\pi xf\ (\sin x)\ dx$ का मान है–

(A*) $\frac{\pi}{2} \int_0^\pi f\ (\sin x)\ dx$ (B*) $\pi \int_0^{\pi/2} f\ (\sin x)\ dx$ (C) 0 (D) $2\pi \int_0^{\pi/4} f\ (\sin x)\, dx\,]$

Sol. $I = \int_0^\pi x\ \ f(\sin x)\ \ dx = \int_0^\pi (\pi - x) f(\sin x)\ \ dx\ = \ \pi \int_0^{\pi/2} f(\sin x)\ \ dx$

5. If $I = \int_0^{2\pi} \sin^2 xdx$, then **[16JM120579]**

(A*) $I = 2\int_0^{\pi} \sin^2 xdx$ (B*) $I = 4\int_0^{\pi/2} \sin^2 xdx$ (C*) $I = \int_0^{2\pi} \cos^2 xdx$ (D) $I = 8\int_0^{\pi/4} \sin^2 xdx$

यदि $I = \int_0^{2\pi} \sin^2 xdx$, हो, तो –

(A*) $I = 2\int_0^{\pi} \sin^2 xdx$ (B*) $I = 4\int_0^{\pi/2} \sin^2 xdx$ (C*) $I = \int_0^{2\pi} \cos^2 xdx$ (D) $I = 8\int_0^{\pi/4} \sin^2 xdx$

Sol. $I = 2\int_0^{\pi} \sin^2 x\ \ dx = 2\ .\ 2\int_0^{\pi/2} \sin^2 x\ \ dx$

6.✍ Given f is an odd function defined everywhere, periodic with period 2 and integrable on every interval. Let

$g(x) = \int_0^x f(t)\ dt$. Then :

(A*) g(2n) = 0 for every integer n
(B*) g(x) is an even function
(C*) g(x) and f(x) have the same period
(D) g(x) is an odd function

दिया गया है कि एक विषम फलन f प्रत्येक बिन्दु पर परिभाषित है, आवर्तकाल 2 वाला आवर्तीफलन है और प्रत्येक अन्तराल में समाकलनीय है। माना $g(x) = \int_0^x f(t)\ dt$ हो, तो–

(A) प्रत्येक पूर्णांक n के लिए g(2n) = 0
(B) g(x) एक सम फलन है।
(C) g(x) और f(x) के आवर्तकाल समान है।
(D) g(x) एक विषम फलन है।

Sol. $f(-x) = -f(x)$ ... (1)
$f(x + 2) = f(x)$ ... (2)

$g(2n) = \int_0^{2n} f(t)\ \ dt = n\int_0^2 f(t)\ \ dt \quad \Rightarrow \quad g(2n) = n\,g(2)$ ... (3)

Now $g(-x) = \int_0^{-x} f(t)dt$

put $t = -z \ \Rightarrow\ dt = -dz = \int_0^x f(-z)\ \ (-dz) = -\int_0^x f(-z)\ \ dz$ (from (1))

$= \int_0^x f(t)\ \ dt = g(x) \qquad \therefore \qquad g(-x) = g(x)$

Again $g(x + 2) = \int_0^x f(t)dt + \int_x^{x+2} f(t)dt \quad \therefore \quad g(x+2) = \int_0^x f(t)\ \ dt + \int_0^2 f(t)\ \ dt$ $(\because\ \ f \to \text{period})$

$\Rightarrow \quad g(x + 2) = g(x) + g(2)$ .... (4)
Putting x = 0, 2, ...
$g(2) = g(0) + g(2) \ \Rightarrow\ g(0) = 0$
$g(4) = g(2) + g(2) \ \Rightarrow\ g(4) = 2g(2)$
putting $x \to -x$ we get
$g(2 - x) = g(-x) + g(2) = g(x) + g(2)$
at x = 2
$g(0) = 2g(2) \ \Rightarrow\ g(2) = 0$
$\therefore \quad g(0) = g(\pm 2) = g(\pm 4) = ... = 0$
from (3) g(2n) = 0

& from (4) $g(x + 2) = g(x)$ $\Rightarrow$ prd. of g(x) is 2

7. Let $f : R \to R$ be defined as $f(x) = \int_{-1}^{e^x} \frac{dt}{1+t^2} + \int_{1}^{e^{-x}} \frac{dt}{1+t^2}$, then **[16JM120580]**

(A*) f(x) is periodic

(B*) $f(f(x)) = f(x)\ \forall\ x \in R$

(C) $f(1) = f'(1) = \frac{\pi}{2}$

(D) f(x) is unbounded

Hindi. माना $f : R \to R$ मे परिभाषित फलन $f(x)\ \int_{-1}^{e^x} \frac{dt}{1+t^2} + \int_{1}^{e^{-x}} \frac{dt}{1+t^2}$, तब

(A*) f(x) आवर्ती है।

(B*) $f(f(x)) = f(x)\ \forall\ x \in R$

(C) $f(1) = f'(1) = \frac{\pi}{2}$

(D) f(x) अपरिबद्ध है।

Sol. $f(x) = \int_{-1}^{e^x} \frac{dt}{1+t^2} + \int_{1}^{e^{-x}} \frac{dt}{1+t^2}$

$$f'(x) = \frac{e^x}{1+e^{2x}} - \frac{e^x}{1+e^{2x}} = \frac{e^x}{1+e^{2x}} - \frac{e^x}{1+e^{2x}} = 0$$

Hence f(x) = costant = $f(0) = \int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2}$

अत: f(x) = अचर = $f(0) = \int_{-1}^{1} \frac{dt}{1+t^2} = \frac{\pi}{2}$

8.✍ If $a, b \in R^+$ then $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(k+an)(k+bn)}$ is equal to

(A) $\frac{1}{a-b}\ \ln\frac{b(b+1)}{a(a+1)}$ if $a \neq b$

(B*) $\frac{1}{a-b}\ \ln\frac{a(b+1)}{b(a+1)}$ if $a \neq b$

(C) non existent if a = b

(D*) $\frac{1}{a(1+a)}$ if a = b

यदि $a, b \in R^+$ तब $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(k+an)(k+bn)}$ बराबर है

(A) $\frac{1}{a-b}\ \ln\frac{b(b+1)}{a(a+1)}$ यदि $a \neq b$

(B*) $\frac{1}{a-b}\ \ln\frac{a(b+1)}{b(a+1)}$ यदि $a \neq b$

(C) विद्यमान नही यदि a = b

(D*) $\frac{1}{a(1+a)}$ यदि a = b

Sol. $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{\left(a+\frac{k}{n}\right)\left(b+\frac{k}{n}\right)}\ \int_{0}^{1} \frac{dx}{(a+x)(b+x)}$ = I (say) माना

If यदि a = b

$$I = \int_{0}^{1} \frac{dx}{(a+x)^2} = -\frac{1}{a+x}\Big|_0^1 = -\left(\frac{1}{a+1} - \frac{1}{a}\right) = \frac{1}{a(a+1)}$$

If यदि $a \neq b$

$$I = \frac{1}{a-b}\int_0^1 \left(\frac{1}{b+x} - \frac{1}{a+x}\right)dx = \frac{1}{a-b}\ell n\left(\frac{b+x}{a+x}\right)\Bigg|_0^1$$

$$= \frac{1}{a-b}\ell n\left(\frac{a(b+1)}{b(a+1)}\right)$$

9. Let f(x) = $\int_x^{x+\frac{\pi}{3}} |\sin\theta|\,d\theta$ $(x \in [0, \pi])$ **[16JM120581]**

(A) f(x) is strictly increasing in this interval

(B*) f(x) is differentiable in this interval

(C*) Range of f(x) is $\left[2-\sqrt{3}, 1\right]$

(D*) f(x) has a maxima at x = $\frac{\pi}{3}$

माना f(x) = $\int_x^{x+\frac{\pi}{3}} |\sin\theta|\,d\theta$ $(x \in [0, \pi])$

(A) f(x) इस अन्तराल में निरन्तर वर्धमान है।

(B*) f(x) इस अन्तराल में अवकलनीय है।

(C*) f(x) का परिसर $\left[2-\sqrt{3}, 1\right]$ है।

(D*) f(x), x = $\frac{\pi}{3}$ पर उच्चिष्ठ रखता है।

Sol. Since |sinθ| is continuous f(x) will be differentiable

चूंकि |sinθ| सतत है और f(x) अवकलनीय होगा।

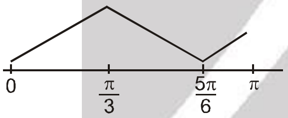
$$f'(x) = \left|\sin\left(x+\frac{\pi}{3}\right)\right| - |\sin x|$$

$$f'(x) = 0 \quad \Rightarrow \quad \sin^2\left(x+\frac{\pi}{3}\right) = \sin^2 x \quad \Rightarrow \quad \sin\left(2x+\frac{\pi}{3}\right).\sin\frac{\pi}{3} = 0$$

$$\Rightarrow \quad 2x + \frac{\pi}{3} = \pi, 2\pi \Rightarrow \quad x = \frac{\pi}{3}, \frac{5\pi}{6}$$

Indicator diagram of f(x) is

f(x) के आरेख से



$$f(0) = \frac{1}{2}$$

$$f\left(\frac{\pi}{3}\right) = 1$$

$$f\left(\frac{5\pi}{6}\right) = 2\int_{5\pi/6}^{\pi} \sin x\,dx = 2-\sqrt{3}$$

$$f(\pi) = \frac{1}{2}$$

10. If f(x) is integrable over [1, 2], then $\int_1^2 f(x)\ dx$ is equal to : **[16JM120578]**

यदि f(x) अन्तराल [1, 2] में समाकलनीय है, तब $\int_1^2 f(x)\,dx$ का मान है–

(A) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} f\left(\frac{r}{n}\right)$ (B*) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=n+1}^{2n} f\left(\frac{r}{n}\right)$

(C*) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} f\left(\frac{r+n}{n}\right)$ (D) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{2n} f\left(\frac{r}{n}\right)$

Sol. (A) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$

(B) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1+n}^{2n} f\left(\frac{r}{n}\right) = \int_1^2 f(x)dx$

$\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{2n} f\left(\frac{r}{n}\right) = \int_0^2 f(x)\ dx$

(C) $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{r=1}^{n} f\left(\frac{r+n}{n}\right) = \int_0^1 f(1+x)\ dx = \int_1^2 f(t)\ dt = \int_1^2 f(x)\ dx$

11. Let माना $I_n = \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^n}}dx$ where जहाँ n > 2 , then तब

(A*) $I_n < \frac{\pi}{6}$ (B) $I_n > \frac{\pi}{6}$ (C) $I_n < \frac{1}{2}$ (D*) $I_n > \frac{1}{2}$

Sol. If यदि n > 2

$x^n < x^2 \quad \forall x \in \left(0, \frac{1}{2}\right) \quad \Rightarrow \quad \sqrt{1-x^n} > \sqrt{1-x^2} \quad \forall x \in \left(0, \frac{1}{2}\right]$

$\Rightarrow 1 < \frac{1}{\sqrt{1-x^n}} < \frac{1}{\sqrt{1-x^2}} \quad = \frac{1}{2} < I_n < \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6}$

Hence अत: (A) & (D)

12. If f(x) = $2^{\{x\}}$, where {x} denotes the fractional part of x. Then which of the following is true ?
[16JM120582]

(A*) f is periodic (B*) $\int_0^1 2^{\{x\}}dx = \frac{1}{\ell n2}$ (C*) $\int_0^1 2^{\{x\}}dx = \log_2 e$ (D*) $\int_0^{100} 2^{\{x\}}dx = 100\log_2 e$

यदि f(x) = $2^{\{x\}}$, जहाँ {x} x के भिन्नात्मक भाग को प्रदर्शित करता है। तब निम्न में कौनसा सत्य है?

(A*) f आवर्ती फलन है (B*) $\int_0^1 2^{\{x\}}dx = \frac{1}{\ell n2}$ (C*) $\int_0^1 2^{\{x\}}\ dx = \log_2 e$ (D*) $\int_0^{100} 2^{\{x\}}\ dx = 100\log_2 e$

Sol. f(x) = $2^{\{x\}}$

Clearly f(x) is periodic with period 1.

Now $\int_0^1 2^{\{x\}}dx = \int_0^1 2^x dx = \left[\frac{2^x}{\ell n2}\right]_0^1 = \frac{1}{\ell n2} = \log_2 e$

Also $\int_0^{100} 2^{\{x\}}dx = 100\int_0^1 2^{\{x\}}dx = 100\ \log_2 e$

Hindi. f(x) = $2^{\{x\}}$

स्पष्टतया f(x) आवर्त 1 का आवर्ती फलन है।

अब $\int_0^1 2^{\{x\}}dx = \int_0^1 2^x dx = \left[\frac{2^x}{\ell n2}\right]_0^1 = \frac{1}{\ell n2} = \log_2 e$

तथा $\int_0^{100} 2^{\{x\}}dx = 100\int_0^1 2^{\{x\}}dx = 100\log_2 e$

$\Big[$ $\int_0^{na} f(x)\ \ dx = n\int_0^a f(x)\ \ dx$ का प्रयोग करने पर यदि $f(x)$का आवर्तकाल a हो $\Big]$

13. Let $f(x) = \int_0^x |2t-3|\,dt$, then f is

(A*) continuous at x = 3/2 (B*) continuous at x = 3

(C*) differentiable at x = 3/2 (D*) differentiable at x = 0

माना $f(x) = \int_0^x |2t-3|\,dt$, तब f है –

(A*) x = 3/2 पर सतत् (B*) x = 3 पर सतत्

(C*) x = 3/2 पर अवकलनीय (D*) x = 0 पर अवकलनीय

Sol. If (यदि) $x \le \frac{3}{2}$

$f(x) = \int_0^x (3-2t)\ \ dt\ \ = 3x - x^2$

$x > \frac{3}{2}$

$f(x) = \int_0^{3/2} (3-2t)\ \ dt + \int_{3/2}^x (2t-3)\ \ dt$

$= \frac{9}{2} + x^2 - 3x$

$$f(x) = \begin{cases} 3x - x^2 & , \quad x \le 3/2 \\ x^2 - 3x + 9/2 & , \quad x > 3/2 \end{cases}$$

Now this is continuous at $x = \frac{3}{2}$ and at x = 3 also differentiable at x = 0

अब यह $x = \frac{3}{2}$ तथा x = 3 पर सतत् है और x = 0 पर अवकलनीय भी है।

14. Let $I_n = \int_0^{\pi} (\sin x)^n dx,\ n \in N$, then **[16JM120583]**

(A*) I_n is rational if n is odd (B*) I_n is irrational if n is even

(C) I_n is an increasing sequence (D*) I_n is a decreasing sequence

माना $I_n = \int_0^{\pi} (\sin x)^n dx,\ n \in N$, तब

(A*) I_n परिमेय है यदि n विषम है। (B*) I_n अपरिमेय है यदि n सम है।

(C) I_n वर्धमान अनुक्रम है। (D*) I_n ह्रासमान अनुक्रम है।

Sol. $I_n = \int_0^{\pi} (\sin x)^n dx = \int_0^{\pi/2} (\sin x)^n dx$

By Wallis' formula we knew that I_n is irrational of n is even & rational if n is odd.

Morever $(\sin x)^n \geq (\sin x)^{n+1}$ $\forall x \in \left[0, \frac{\pi}{2}\right]$

Hence $I_n > I_{n+1}$, so it is a decreasing sequence.

Hindi. $I_n = \int_0^{\pi} (\sin x)^n dx = \int_0^{\pi/2} (\sin x)^n dx$

वॉली सूत्र से हम जानते है कि I_n अपरिमेय है यदि n सम है और परिमेय यदि n विषम है।

यहाँ $(\sin x)^n \geq (\sin x)^{n+1}$ $\forall x \in \left[0, \frac{\pi}{2}\right]$ अतः $I_n > I_{n+1}$, इसलिए यह ह्रासमान अनुक्रम है।

15.✍ Let f(x) be a function satisfying f(x) + f(x + 2) = 10 $\forall x \in R$, then

(A*) f(x) is a periodic function (B) f(x) is aperiodic function

(C) $\int_{-1}^{7} f(x)dx = 20$ (D*) $\int_{-1}^{7} f(x)dx = 40$

माना f(x) एक फलन f(x) + f(x + 2) = 10 $\forall x \in R$, को सन्तुष्ट करता है तब f(x)

(A*) f(x) आवर्ती फलन है। (B) f(x) आवर्ती फलन नही है।

(C) $\int_{-1}^{7} f(x)dx = 20$ (D*) $\int_{-1}^{7} f(x)dx = 40$

Sol. f(x) + f(x + 2) = 10 ......(i)

replace x by x + 2 x को x + 2 से हटाने पर

f(x + 2) + f(x + 4) = 10 ....(ii) (i) – (ii) $\Rightarrow$ f(x) – f(x + 4) = 0

Now अब $\int_{-1}^{7} f(x)dx = 2\int_0^4 f(x)dx = 2\int_0^2 (f(x) + f(x+2))dx = 2\int_0^2 10dx = 2 \cdot 2 \cdot 10 = 40$

Hence अतः (A) & (D)

16. Let $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx$, $n \in N$, then **[16JM120584]**

(A*) $I_{n+2} + I_n = 2I_{n+1}$ (B) $I_n = I_{n+1}$

(C*) $I_n = n\pi$ (D) $I_1, I_2, I_3, ...I_n$ are in Harmonic progression

माना $I_n = \int_0^{\pi} \frac{\sin^2(nx)}{\sin^2 x} dx$, $n \in N$, तब

(A*) $I_{n+2} + I_n = 2I_{n+1}$ (B) $I_n = I_{n+1}$

(C*) $I_n = n\pi$ (D) $I_1, I_2, I_3, ...I_n$ हरात्मक श्रेणी मे है।

Sol. Consider माना कि $I_{n+2} - 2\, I_{n+1} + I_n = \int_0^{\pi} \frac{\sin^2(n+2)x - 2\sin^2(n+1)x + \sin^3(nx)}{\sin^2 x} dx$

$$= \int_0^{\pi} \frac{\{\sin^2(n+2)x - \sin^2(n+1)x\}\{\sin^2(n+1)x - \sin^2 nx\}}{\sin^2 x} dx = \int_0^{\pi} \frac{\sin x\{\sin(2n+3)x - \sin(2n+1)x\}}{\sin^2 x} dx$$

$$= \int_0^{\pi} \frac{2\cos(2x+2)\sin x}{\sin x} dx = \frac{1}{n+1}\sin(2n+3)x\Big|_0^{\pi} = 0$$

Hence I_1, I_1,I_2, ... are in A.P. अतः I_1, I_1,I_2, ... समान्तर श्रेणी मे है।

$$I_1 = \int_0^{\pi} \frac{\sin^2 x}{\sin^2 x} dx = \pi \ , \ I_2 = \int_0^{\pi} \frac{4\sin^2 x\cos^2 x}{\sin^2 x} dx = 2\pi$$

Hence अतः $I_n = \pi + (n - 1)\pi = n\pi$

$$\int_1^{100} \frac{f(x)}{x} dx = \int_1^{10} \frac{f(x)}{x} dx + \int_{10}^{100} \frac{f(x)}{x} dx = 5 + \int_{10}^{1} \frac{f\left(\frac{100}{t}\right)}{100}.t\left(\frac{-100dt}{t^2}\right)$$ (substituting रखने पर $x = \frac{100}{t}$)

$$= 5 + \int_t^{10} f\left(\frac{100}{t}\right).\frac{dt}{t} = 5 + \int_1^{10} \frac{f(t)}{t} dt = 10$$

17. Let f(x) be a continuous function and

$$I = \int_1^9 \sqrt{x}\, f(x)\ dx \text{ , then}$$

(A*) There exists some c ∈ (1, 9) such that $I = 8\sqrt{c}\, f(c)$

(B*) There exists some p, q ∈ (1, 3) such that $I = 2[p^2\ f(p^2) + q^2\ f(q^2)]$

(C) There exists some α ∈ (1, 9) such that $I = 9\sqrt{\alpha}\, f(\alpha)$

(D) If $f(x) \geq 0\ \forall x \in [1, 9] \quad \Rightarrow \quad I > 0$

माना कि f(x) सतत फलन है और

$$I = \int_1^9 \sqrt{x}\, f(x)\ dx \text{ , तब}$$

(A*) कोई c ∈ (1, 9) इस प्रकार विद्यमान है कि $I = 8\sqrt{c}\, f(c)$

(B*) कोई p, q ∈ (1, 3) इस प्रकार विद्यमान है कि $I = 2[p^2\ f(p^2) + q^2\ f(q^2)]$

(C) कोई α ∈ (1, 9) इस प्रकार विद्यमान है कि $I = 9\sqrt{\alpha}\, f(\alpha)$

(D) यदि $f(x) \geq 0\ \forall x \in [1, 9] \quad \Rightarrow \quad I > 0$

Sol. Since f(x) is continuous

$$I = \int_1^9 \sqrt{x} + f(x) = \sqrt{c}.f(c)(9-1) \qquad \text{for some } c \in (1, 9)$$

$= 8\sqrt{c}\ f(c)$

Now substituting $x = t^2$.

$$I = 2\int_1^3 t^2 f(t^2)dt \;=\; 2\left[\int_1^2 t^2 f(t^2)\,dt + \int_2^3 t^3 f(t^2)\,dt\right] = 2[p^2f(p^2) + q^2f(q^2)] \text{ for some } p \in (1, 2) \;\&\; q \in (1, 3)$$

Hence (A), (B) are correct option.

Sol. चूंकि f(x) सतत है।

$$I = \int_1^9 \sqrt{x} + f(x) = \sqrt{c}.f(c)(9-1) \quad \text{किसी } c \in (1, 9) \text{ के लिए}$$

$= 8\sqrt{c}\ f(c)$

अब $x = t^2$ रखने पर.

$$I = 2\int_1^3 t^2 f(t^2)dt \;=\; 2\left[\int_1^2 t^2 f(t^2)\,dt + \int_2^3 t^3 f(t^2)\,dt\right] = 2[p^2f(p^2) + q^2f(q^2)] \text{ किसी p के लिए} \in (1, 2) \;\&\; q \in (1, 3)$$

अतः (A) और (B) विकल्प सही है।

18. Let माना $A = \int_1^{e^2} \frac{\ell nx}{\sqrt{x}}dx$, then तब

(A) $A > 2\left(e - \frac{1}{e}\right)$

(B*) $A < (e-1)\left(2 + \frac{1}{\sqrt{e}}\right)$

(C) $A > (e-1)\left(2 + \frac{1}{\sqrt{e}}\right)$

(D*) $A < \left(e^2 - 1\right)\frac{2}{e}$

Sol. Let माना $f(x) = \frac{\ell nx}{\sqrt{x}}$

$f'(x) = \frac{2 - \ell nx}{2x^{3/2}} = 0 \Rightarrow x = e^2$

f(x) is decreasing in (e^2, ∞) and f(x) is increasing in $(0,e^2)$

f(x), (e^2, ∞) में हासमान है तथा f(x), $(0,e^2)$ में वर्धमान है

$A = \int_1^{e^2} \frac{\ell nx}{\sqrt{x}}dx < (e^2 - 1)\ f(e^2) \quad \Rightarrow \quad A < (e^2 - 1)\ \frac{2}{e}$

$A = \int_1^{e^2} \frac{\ell nx}{\sqrt{x}} = \int_1^{e} \frac{\ell nx}{\sqrt{x}} + \int_e^{e^2} \frac{\ell nx}{\sqrt{x}}dx$

$A < (e-1)\ \frac{1}{\sqrt{e}} + (e^2 - e)\ \frac{2}{e} \quad \Rightarrow \quad A < (e-1)\left(2 + \frac{1}{\sqrt{e}}\right)$

19. Let $f(a, b) = \int_a^b (x^2 - 4x + 3)dx$, $(b > a)$ then **[16JM120585]**

(A*) f(a, 3) is least when a = 1

(B*) f(4, b) is an increasing function $\forall b \geq 4$

(C) f(0, b) is least for b = 2 (D*) min{f(a, b)} = $-\frac{4}{3}$

माना $f(a,b) = \int_a^b (x^2 - 4x + 3)dx$, (b > a), तब

(A*) f(a,3) न्यूनतम है जब a = 1 (B*) f(4,b) वर्धमान फलन है $\forall b \geq 4$

(C) b = 2 के लिए f(0, b) न्यूनतम है (D*) min{f(a, b)} = $-\frac{4}{3}$

Sol. $f(a, b) = \int_a^b (x^2 - 4x + 3)dx \Rightarrow f(a, b) = \left[\frac{x^3}{3} - 2x^2 + 3x\right]_a^b = \frac{b^3 - a^3}{3} - 2(b^2 - a^2) + 3(b - a)$

20. Let $I = \int_2^\infty \left(\frac{\lambda x}{x^2+1} - \frac{1}{2x+1}\right) dx$ & I is a finite real number, then

माना $I = \int_2^\infty \left(\frac{\lambda x}{x^2+1} - \frac{1}{2x+1}\right)$ dx तथा I एक निश्चित वास्तविक संख्या है तब

(A*) $\lambda = \frac{1}{2}$ (B) $\lambda = 1$ (C) $I = \frac{1}{2}\ell n\left(\frac{5}{2}\right)$

(D*) $I = \frac{1}{4}\ell n\left(\frac{5}{4}\right)$

Sol. $I = \lim_{t\to\infty} \frac{\pi}{2}\ell n(x^2+1) - \frac{1}{2}\ell n(2x+1)\Big|_2^t$; $I = \lim_{t\to\infty} \frac{1}{2}\ell n\left(\frac{(t^2+1)^\lambda}{2t+1}\right) - \frac{\lambda}{2}\ell n5 + \frac{1}{2}\ell n5$

If I is finite $\lim_{t\to\infty} \frac{(t^2+1)^\lambda}{2t+1}$ must approach a finite positive number.

Hence $\lambda = \frac{1}{2}$ & $\lim_{t\to\infty} \frac{(t^2+1)^{1/2}}{2t+1} = \frac{1}{2}$

so $I = \frac{1}{2}\ell n\left(\frac{1}{2}\right) - \frac{1}{4}\ell n5 + \frac{1}{2}\ell n5$

Hence (A) & (D) are correct option.

Hindi $I = \lim_{t\to\infty} \frac{\pi}{2}\ell n(x^2+1) - \frac{1}{2}\ell n(2x+1)\Big|_2^t$; $I = \lim_{t\to\infty} \frac{1}{2}\ell n\left(\frac{(t^2+1)^\lambda}{2t+1}\right) - \frac{\lambda}{2}\ell n5 + \frac{1}{2}\ell n5$

यदि I निश्चित है $\lim_{t\to\infty} \frac{(t^2+1)^\lambda}{2t+1}$ एक धनात्मक संख्या की ओर जाता है।

अतः $\lambda = \frac{1}{2}$ & $\lim_{t\to\infty} \frac{(t^2+1)^{1/2}}{2t+1} = \frac{1}{2}$

इसलिए $I = \frac{1}{2}\ell n\left(\frac{1}{2}\right) - \frac{1}{4}\ell n5 + \frac{1}{2}\ell n5$

अतः (A) तथा (D) सही विकल्प है।

21. Let f(x) be a strictly increasing, non-negative function such that $f''(x) < 0 \ \forall x \in (\alpha, \beta)$ & $I = \int\limits_{\alpha}^{\beta} f(x)dx$ $(\beta > \alpha)$, then **[16JM120586]**

माना f(x) निरन्तर वर्धमान, अऋणात्मक फलन इस प्रकार है कि $f''(x) < 0 \ \forall x \in (\alpha, \beta)$ & $I = \int\limits_{\alpha}^{\beta} f(x)dx$ $(\beta > \alpha)$, तब

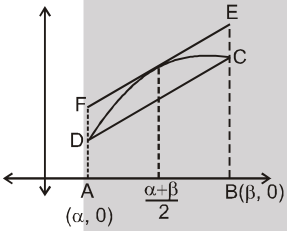
(A*) $I < f\left(\frac{\alpha+\beta}{2}\right)(\beta-\alpha)$

(B) $I > f\left(\frac{\alpha+\beta}{2}\right)(\beta-\alpha)$

(C*) $I > \frac{1}{2}(f(\alpha) + f(\beta))(\beta - \alpha)$

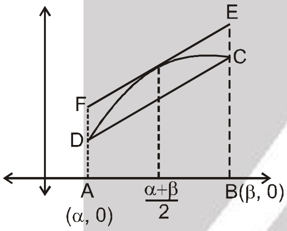
(D) $I < \frac{1}{2}(f(\alpha) + f(\beta))(\beta - \alpha)$

Sol. Now (Area of trapezium ABCD) < I < (Area of trapezium ABEF)



$$\frac{1}{2}(f(\alpha) + f(\beta)(\beta - \alpha)) < I < \frac{1}{2}\left\{2f\left(\frac{\alpha+\beta}{2}\right)\right\}(\beta-\alpha)$$

Hindi. अब (ABCD समलम्ब चतुर्भुज का क्षेत्रफल) < I < (Area of trapezium ABEF)



$$\frac{1}{2}(f(\alpha) + f(\beta)(\beta - \alpha)) < I < \frac{1}{2}\left\{2f\left(\frac{\alpha+\beta}{2}\right)\right\}(\beta-\alpha)$$

22. $I_1 = \int\limits_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$, $I_2 = \int\limits_{0}^{\pi} \frac{x^3 \sin x}{(\pi^2 - 3\pi x + 3x^2)(1+\cos^2 x)}dx$, then तब

(A) $I_1 = \frac{\pi^2}{8}$

(B*) $I_1 = \frac{\pi^2}{4}$

(C*) $I_1 = I_2$

(D) $I_1 > I_2$

Sol. $I_1 = \int\limits_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \int\limits_{0}^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$

$2\,I_1 = \pi \int\limits_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = I_1 = \pi \int\limits_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$

$$I_2 = \int_0^{\pi} \frac{x^3 \sin x}{(\pi^2 - 3\pi x + 3x^2)(1+\cos^2 x)}dx = \int_0^{\pi} \frac{(\pi - x)^3 \sin x dx}{\{\pi^2 - 3\pi(\pi - x) + 3(\pi - x)^2(1+\cos^2 x)}$$

$$I_2 = \int_0^{\pi} \frac{(\pi - x)^3 \sin x}{\{\pi^2 - 3\pi x + 3x^2\}(1+\cos^2 x)}dx = \int_0^{\pi} \frac{\pi(\pi^2 - 3\pi x + 3x^2)\sin x}{(\pi^2 - 3\pi x + 3x^2\}(1+\cos^2 x)}dx$$

$$\Rightarrow \qquad 2I_2 = \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx \qquad \Rightarrow \qquad I_1 = I_2$$

23. Let $L_1 = \lim\limits_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x - \sin x}$, $L_2 = \lim\limits_{x\to 0^-} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x - \sin x}$, then identify the correct option(s). **[16JM120587]**

माना $L_1 = \lim\limits_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x - \sin x}$, $L_2 = \lim\limits_{x\to 0^-} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x - \sin x}$, तब सही कथन/कथनों को बताइये।

(A*) $L_1 = 4$ (B) $L_1 + L_2 = 8$

(C*) $L_1 + L_2 = 0$ (D*) $|L_2| = |L_1|$

Sol. $L_1 = \lim\limits_{x\to 0^+} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x - \sin x} = \lim\limits_{x\to 0^+} \frac{\sin(x).2x}{1 - \cos x}$ (By L'H'opitals rule) (L हॉस्पीटल नियम से)

$= 4$

$L_2 = \lim\limits_{x\to 0^-} \frac{\int_0^{x^2} \sin\sqrt{t}\,dt}{x - \sin x} = \lim\limits_{x\to 0^-} \frac{\sin(-x).2x}{1 - \cos x}$ (By L'H'opitals rule) (L हॉस्पीटल नियम से)

$= -4$

Hence (A), (C) & (D) are correct option. अतः (A), (C), (D) विकल्प सही है।

24. ✍ $\lim\limits_{n\to\infty} \frac{(1^k + 2^k + 3^k + ... + n^k)}{(1^2 + 2^2 + ... + n^2)(1^3 + 2^3 + ... + n^3)} = F(k)$, then $(k \in N)$

(A*) F(k) is finite for $k \le 6$ (B*) $F(5) = 0$

(C*) $F(6) = \frac{12}{7}$ (D) $F(6) = \frac{5}{7}$

Hindi $\lim\limits_{n\to\infty} \frac{(1^k + 2^k + 3^k + ... + n^k)}{(1^2 + 2^2 + ... + n^2)(1^3 + 2^3 + ... + n^3)} = F(k)$, तब $(k \in N)$

(A*) F(k) निश्चित है $k \le 6$ के लिए (B*) $F(5) = 0$

(C*) $F(6) = \frac{12}{7}$ (D) $F(6) = \frac{5}{7}$

Sol. $$\lim_{n\to\infty}\frac{(1^k+2^k+3^k+.....+n^k)}{(1^2+2^2+.....+n^2)(1^3+2^3+........+n^3)}$$

Now अब $\lim\limits_{n\to\infty}\dfrac{1^k+2^k+......n^k}{n^{k+1}} = \displaystyle\int_0^1 x^k dx = \dfrac{1}{k+1}$ $\forall\ p \in N$

Hence the given limit is finite of $k \le 6$ & equals 12/7 for k = 6.

अतः दी गई सीमा निश्चित है $k \le 6$ जो 12/7 के बराबर है। k = 6 के लिए।

25. Let माना $T_n = \displaystyle\sum_{r=1}^{n}\frac{n}{r^2-2r.n+2n^2}$, $S_n = \displaystyle\sum_{r=0}^{n-1}\frac{n}{r^2-2r.n+2n^2}$, then तब **[16JM120588]**

(A*) $T_n > S_n\ \forall\ n \in N$

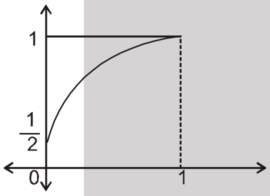
(B*) $T_n > \frac{\pi}{4}$

(C*) $S_n < \frac{\pi}{4}$

(D*) $\lim\limits_{n\to\infty} S_n = \frac{\pi}{4}$

Sol. $T_n = \dfrac{1}{n}\displaystyle\sum_{r=1}^{n}\frac{1}{\left(\frac{r}{n}\right)^2-2\left(\frac{r}{n}\right)+2}$ $\qquad S_n = \dfrac{1}{n}\displaystyle\sum_{r=0}^{n-1}\frac{1}{\left(\frac{r}{n}\right)^2-2\left(\frac{r}{n}\right)+2}$

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} S_n = \int_0^1 \frac{dx}{x^2-2x+2} = \tan^{-1}(x-1)\ \Big|_0^1 = \frac{\pi}{4}$$



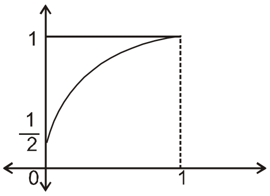
Note that $f(x) = \dfrac{1}{x^2-2x+2}$ is increasing in [0, 1] hence

$T_n > S_n\ \forall\ n \in N$

and T_n is a decreasing sequence, while S_n is an increasing sequence. Hence $T_n > \lim\limits_{n\to\infty} T_n$ & $S_n < \lim\limits_{n\to\infty} S_n$ so $T_n > \frac{\pi}{4}$ & $S_n < \frac{\pi}{4}$

Hindi. $T_n = \dfrac{1}{n}\displaystyle\sum_{r=1}^{n}\frac{1}{\left(\frac{r}{n}\right)^2-2\left(\frac{r}{n}\right)+2}$ $\qquad S_n = \dfrac{1}{n}\displaystyle\sum_{r=0}^{n-1}\frac{1}{\left(\frac{r}{n}\right)^2-2\left(\frac{r}{n}\right)+2}$

$$\lim_{n\to\infty} T_n = \lim_{n\to\infty} S_n = \int_0^1 \frac{dx}{x^2-2x+2} = \tan^{-1}(x-1)\ \Big|_0^1 = \frac{\pi}{4}$$



दिया गया है $f(x) = \dfrac{1}{x^2 - 2x + 2}$ [0, 1] मे वर्धमान है। अतः

$T_n > S_n \ \forall \ n \in N$

तथा T_n वर्धमान अनुक्रम है जबकि S_n वर्धमान अनुक्रम है।

$T_n > \lim\limits_{n \to \infty} T_n$ तथा $S_n < \lim\limits_{n \to \infty} S_n$ इसलिए $T_n > \dfrac{\pi}{4}$ तथा $S_n < \dfrac{\pi}{4}$

26. $f(x) = \int_0^1 f(tx)dt$, where f ' (x) is a continuous function such that f(1) = 2, then

(A*) f(x) is a periodic function (B*) f '(x) = 0

(C*) f(x) is an even function (D) f(x) is an odd function

$f(x) = \int_0^1 f(tx)dt$, जहाँ f ' (x) सतत फलन इस प्रकार है कि f(1) = 2, तब

(A*) f(x) आवर्ती फलन है। (B*) f '(x) = 0

(C*) f(x) समफलन है। (D) f(x) विषम फलन है।

Sol. $f(x) = \int_0^1 f(tx)dt$ माना $tx = y \Rightarrow xdt = dy$

$f(x) = \dfrac{1}{x}\int_0^x f(y).dt \Rightarrow xf(x) = \int_0^x f(y)dy \Rightarrow xf'(x) + f(x) = f(x)$

$\Rightarrow$ $f'(x) = 0 \ \ \forall x \in R$ चूंकि f ' (x) सतत है

$\Rightarrow$ f(x) = 2 (क्योकि f(1) = 2). अतः (A), (B), (C)

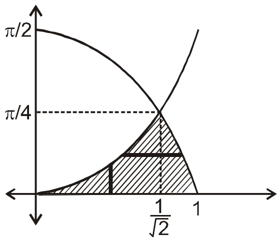
27. Area bounded by $y = \sin^{-1}x$, $y = \cos^{-1}x$, y = 0 in first quadrant is equal to : **[16JM120589]**

$y = \sin^{-1}x$, $y = \cos^{-1}x$, y = 0 से परिबद्ध क्षेत्रफल प्रथम चतुर्थांश मे बराबर है

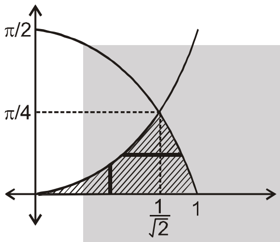
(A*) $\int_0^{1/\sqrt{2}} (\sin^{-1}x)dx + \int_{1/\sqrt{2}}^{1} (\cos^{-1}x)dx$ (B*) $\int_{\pi/4}^{\pi/2} (\sin y - \cos y)dy$

(C*) $\int_0^{\pi/4} (\cos y - \sin y)dy$ (D*) $(\sqrt{2} - 1)$ sq.unit वर्ग इकाई

Sol. Shaded area can be expressed by any one of A, B, C & it equals $\sqrt{2}-1$. Hance (A), (B), (C) & (D) are correct option.



Hindi A, B, C मे से किसी एक से व्यक्त करने वाला भाग $\sqrt{2}-1$. है। अतः (A), (B), (C) व (D) सही विकल्प है।



28. Let f(x) be a non-negative, continuous and even function such that area bounded by x-axis, y-axis & y = f(x) is equal to $(x^2 + x^3)$ sq. units $\forall x \geq 0$, then

माना f(x) अऋणात्मक सतत् और सम फलन इस प्रकार है कि x-अक्ष, y-अक्ष, y = f(x) से परिबद्ध क्षेत्रफल $(x^2 + x^3)$ वर्ग इकाई है $\forall x \geq 0$, तब

(A*) $\sum_{r=1}^{n} f'(r) = 3n^2 + 5n \ \forall \ n \in N$

(B) $\sum_{r=1}^{n} f'(r) = 6n^2 + 5n \ \forall \ n \in N$

(C) $f(x) = 3x^2 + 2x \ \forall \ x \leq 0$

(D*) $f(x) = 3x^2 - 2x \ \forall \ x \leq 0$

Sol. $A = \int_0^x f(t)dt = (x^2 + x^3) \quad \forall \ x \geq 0 \quad \Rightarrow \quad f(x) = 2x + 3x^2 \quad \forall \ x \geq 0$

as f(x) is even चूंकि f(x) सम है

$f(x) = 3x^2 - 2x \quad \forall \ x \geq 0$

now अब $f'(x) = 6x + 2 \ \ \forall \ x \geq 0$

$$\sum_{r=1}^{n} f'(r) = 6.\frac{n(n+1)}{2} + 2n \ = 3n^2 + 5n$$

29. Let 'c' be a positive real number such that area bounded by y = 0 y = $[\tan^{-1}x]$ from x = 0 to x = c is equal to area bounded by y = 0, y = $[\cot^{-1} x]$, from x = 0 to x = c (where [*] represents greatest integer function), then **[16JM120590]**

(A*) c = tan1 + cot1

(B*) c = 2cosec2

(C) c = tan1 – cot1

(D) c = –2 cot2

माना कि 'c' एक धनात्मक वास्तविक संख्या इस प्रकार है कि y = 0 y = $[\tan^{-1}x]$ का x = 0 से x = c से परिबद्ध क्षेत्रफल, वक्र y = 0, y = $[\cot^{-1} x]$, का x = 0 से x = c से परिबद्ध क्षेत्रफल के बराबर है। (जहाँ [*] महत्तम पूर्णांक फलन को व्यक्त करता है।

(A*) c = tan1 + cot1

(B*) c = 2cosec2

(C) $c = \tan 1 - \cot 1$ (D) $c = -2 \cot 2$

Sol. $\int_0^c [\tan^{-1} x]dx = \int_0^c [\cot^{-1} x]dx$

Now clearly $c > \tan 1$ as $[\tan^{-1}x] = 0 \quad \forall\ x \in [0, \tan 1)$

Hence $\int_0^{\tan 1} 0dx + \int_{\tan 1}^{c} 1dx = \int_0^{\cot 1} dx + \int_{\cot 1}^{c} 0dx \quad \Rightarrow \quad 0 + c - \tan 1 = \cot 1 \Rightarrow \tan 1 + \cot 1 = 2\ \text{cosec} 2$

Hindi $\int_0^c [\tan^{-1} x]dx = \int_0^c [\cot^{-1} x]dx$

स्पष्टतया $c > \tan 1$ as $[\tan^{-1}x] = 0$ क्योकि $\forall\ x \in [0, \tan 1)$

अतः $\int_0^{\tan 1} 0dx + \int_{\tan 1}^{c} 1dx = \int_0^{\cot 1} dx + \int_{\cot 1}^{c} 0dx \quad \Rightarrow \quad 0 + c - \tan 1 = \cot 1$

$\Rightarrow \quad \tan 1 + \cot 1 = 2\ \text{cosec} 2$

30. Area bounded by $y = x^2 - 2|x|$ and $y = -1$ is equal to

(A) $2\int_0^1 (2x - x^2)dx$ (B*) $\frac{2}{3}$ sq. units

(C) $\frac{2}{3}$ (Area of rectangle ABCD) where points A, B, C, D are (–1, –1), (–1, 0), (1, 0) & (1, –1)

(D*) $\frac{2}{3}$ (Area of rectangle ABCD) where points A, B, C, D are (–1, –1), (–1, 0), (1, 0) & (1, –1)

Hindi. वक्र $y = x^2 - 2|x|$ और $y = -1$ से परिबद्ध क्षेत्रफल बराबर है।

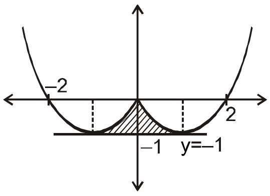
(A) $2\int_0^1 (2x - x^2)dx$ (B*) $\frac{2}{3}$ वर्ग इकाई

(C) $\frac{2}{3}$ (आयत ABCD का क्षेत्रफल जहाँ बिन्दु A, B, C, D क्रमशः (–1, –1), (–1, 0), (1, 0) व (1, –1) है।

(D*) $\frac{2}{3}$ (आयत ABCD का क्षेत्रफल) जहाँ A, B, C, D क्रमशः (–1, –1), (–1, 0), (1, 0) व (1, –1) है।

Sol. Area क्षेत्रफल $2 - 2\int_0^1 (2x - x^2)dx = 2 - 2\left(x^2 - \frac{x^3}{3}\right)\Bigg|_0^1$

$= 2 - \frac{4}{3} = \frac{2}{3}$



hence option (B) & (C) are correct option.

(B) व (C) सही विकल्प है।

PART - IV : COMPREHENSION

भाग - IV : अनुच्छेद (COMPREHENSION)

Comprehension # 1

अनुच्छेद # 1

If $y = \int_{u(x)}^{v(x)} f(t)\ dt$, let us define $\frac{dy}{dx}$ in a different manner as $\frac{dy}{dx} = v'(x)\ f^2(v(x)) - u'(x)\ f^2(u(x))$ and the equation of the tangent at (a, b) as $y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x - a)$

यदि $y = \int_{u(x)}^{v(x)} f(t)\ dt$ है। मानाकि $\frac{dy}{dx}$ को एक भिन्न तरीके $\frac{dy}{dx} = v'(x)\ f^2(v(x)) - u'(x)\ f^2(u(x))$ से भी परिभाषित करते है तथा (a, b) पर स्पर्श रेखा के समीकरण को $y - b = \left(\frac{dy}{dx}\right)_{(a,b)} (x - a)$ से परिभाषित करते है।

1. If $y = \int_{x}^{x^2} t^2\ dt$, then equation of tangent at x = 1 is

यदि $y = \int_{x}^{x^2} t^2\ dt$ है, तो x = 1 पर स्पर्श रेखा का समीकरण है –

(A) y = x + 1 (B) x + y = 1 (C*) y = x – 1 (D) y = x

Sol. At x = 1, y = 0

$\frac{dy}{dx} = 2x\ .\ x^8 - x^4 = 2 - 1 = 1$

∴ equation of tangent y – 0 = 1 (x – 1)

Hindi x = 1 पर y = 0

$\frac{dy}{dx} = 2x\ .\ x^8 - x^4 = 2 - 1 = 1$

∴ स्पर्श रेखा का समीकरण y – 0 = 1 (x – 1)

y = x – 1

2. If $F(x) = \int_{1}^{x} e^{t^2/2} (1 - t^2)\ dt$, then $\frac{d}{dx} F(x)$ at x = 1 is

यदि $F(x) = \int_{1}^{x} e^{t^2/2} (1 - t^2)\ dt$ है, तो x = 1 पर $\frac{d}{dx} F(x)$ है –

(A*) 0 (B) 1 (C) 2 (D) – 1

Sol. $F(x) = \int_{1}^{x} e^{t^2/2} (1 - t^2)\ dt$

$$F'(x) = \left(e^{\frac{x^2}{2}}(1-x^2)\right)$$

$\therefore$ $F'(1) = 0$

3. If $y = \int_{x^3}^{x^4} \ell nt \;\; dt$, then $\lim_{x\to 0^+} \frac{dy}{dx}$ is

यदि $y = \int_{x^3}^{x^4} \ell nt \;\; dt$ है, तो $\lim_{x\to 0^+} \frac{dy}{dx}$ है –

(A*) 0 (B) 1 (C) 2 (D) – 1

Sol. $\frac{dy}{dx} = 4x^3 (\ell n\, x^4)^2 - 3x^2 (\ell n\, x^3)^2$

$= 4x^3 (4\, \ell n\, x)^2 - 3x^2 (3\, \ell n\, x)^2$

$= 64x^3 (\ell n\, x)^2 - 27\, x^2 (\ell n\, x)^2$

$\therefore$ $\lim_{x\to 0^+} \frac{dy}{dx} = 64 \lim_{x\to 0^+} x^3 (\ell n\, x)^2 - 27 \lim_{x\to 0^+} x^2 (\ell n\, x)^2 = 0$

Comprehension # 2

Let $g(t) = \int_{x_1}^{x_2} f(t,\;\; x) \;\; dx$. Then $g'(t) = \int_{x_1}^{x_2} \frac{\partial}{\partial t}$ (f(t, x)) dx. Consider $f(x) = \int_0^{\pi} \frac{\ell n \;\; (1 + x\cos\theta)}{\cos\theta} d\theta$.

मानाकि $g(t) = \int_{x_1}^{x_2} f(t,\;\; x) \;\; dx$ हो, तो $g'(t) = \int_{x_1}^{x_2} \frac{\partial}{\partial t}$ (f(t, x)) dx तथा माना $f(x) = \int_0^{\pi} \frac{\ell n \;\; (1 + x\cos\theta)}{\cos\theta} d\theta$ है।

4. Range of f(x) is **[16JM120591]**

f(x) का परिसर है –

(A) $(0, \pi)$ (B) $(0, \pi^2)$ (C) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (D*) $\left(\frac{-\pi^2}{2}, \frac{\pi^2}{2}\right)$

5. The number of critical points of f(x), in the interior of its domain, is **[16JM120592]**

(A*) 0 (B) 1 (C) 2 (D) infinitely many

f(x) के प्रांत में स्थित क्रांतिक बिन्दुओं की संख्या है –

(A*) 0 (B) 1 (C) 2 (D) अनन्त

6. f(x) is **[16JM120593]**

(A) discontinuous at x = 0 (B*) differentiable at x = 1

(C*) continuous at x = 0 (D) None of these

f(x) है–

(A) x = 0 पर असतत् (B*) x = 1 पर अवकलनीय

(C*) x = 0 पर सतत् (D) इनमे से कोई नहीं

Sol. (4, 5, 6)

For the integral to be defined in $(0, \pi)$, $-1 < x < 1$ **[T]**

$$f'(x) = \int_0^{\pi} \frac{1}{1 + x\cos\theta} \;\; d\theta \;\; = \int_0^{\pi} \frac{1}{1 - x\cos\theta} \;\; d\theta \qquad \left(\because \;\; \int_0^{\pi} g \;\; (\theta) \;\; d\theta = \int_0^{\pi} g \;\; (\pi - \theta) \;\; d\theta\right)$$

$$\Rightarrow \qquad 2f'(x) = \int_0^{\pi} \frac{2}{1 - x^2\cos^2\theta} \;\; d\theta$$

$$f'(x) = \int_0^{\pi} \frac{d\theta}{1-x^2\cos^2\theta} = 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{1-x^2\cos^2\theta} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta \quad d\theta}{1+\tan^2\theta-x^2} = 2\int_0^{\infty} \frac{dt}{t^2+1-x^2}$$

$$= 2\,\frac{1}{\sqrt{1-x^2}}\left[\tan^{-1}\frac{t}{\sqrt{1-x^2}}\right]_0^{\infty} = \frac{\pi}{\sqrt{1-x^2}} \qquad \Rightarrow \qquad f(x) = \pi\sin^{-1}x + k$$

but $f(0) = \int_0^{\pi} \frac{\ell n \ \ 1}{\cos\theta}$ $\qquad \therefore \qquad f(x) = \pi\sin^{-1}x$

Range of $f(x) = \left(\frac{-\pi^2}{2}, \ \frac{\pi^2}{2}\right)$ $\quad \left(\because \ \text{range of} \ \sin^{-1}x = \left(\frac{-\pi}{2}, \ \frac{\pi}{2}\right)\right)$

f(x) is differentiable in the interior of its domain and $f'(x) = 0$ has no solution.
Hence f(x) has no critical points.
$f(x) = \pi\sin^{-1}x, \ x \in (-1, 1)$
Applying Lagrange's theorem

$$f'(x) = \frac{f(1)-f(-1)}{1-(-1)}$$

$$\Rightarrow \qquad \frac{\pi}{\sqrt{1-x^2}} = \frac{\pi^2}{2} \qquad \Rightarrow \qquad x = \pm\frac{\sqrt{\pi^2-4}}{\pi} \in (-1, 1)$$

∴ There are two Lagrange's constant for f(x) in its domain.

Hindi **(4, 5, 6)**

समाकलन के $(0, \pi)$ में परिभाषित होने के लिए $-1 < x < 1$ **[T]**

$$f'(x) = \int_0^{\pi} \frac{1}{1+x\cos\theta}\ d\theta = \int_0^{\pi} \frac{1}{1-x\cos\theta}\ d\theta \qquad \left(\because \ \int_0^{\pi} g\ (\theta)\ d\theta = \int_0^{\pi} g\ (\pi-\theta)\ d\theta\right)$$

$$\Rightarrow \qquad 2f'(x) = \int_0^{\pi} \frac{2}{1-x^2\cos^2\theta}\ d\theta$$

$$f'(x) = \int_0^{\pi} \frac{d\theta}{1-x^2\cos^2\theta} = 2\int_0^{\frac{\pi}{2}} \frac{d\theta}{1-x^2\cos^2\theta} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta \quad d\theta}{1+\tan^2\theta-x^2} = 2\int_0^{\infty} \frac{dt}{t^2+1-x^2}$$

$$= 2\,\frac{1}{\sqrt{1-x^2}}\left[\tan^{-1}\frac{t}{\sqrt{1-x^2}}\right]_0^{\infty} = \frac{\pi}{\sqrt{1-x^2}} \qquad \Rightarrow \qquad f(x) = \pi\sin^{-1}x + k$$

लेकिन $f(0) = \int_0^{\pi} \frac{\ell n \ \ 1}{\cos\theta}$ $\qquad \therefore \qquad f(x) = \pi\sin^{-1}x$

f(x) का परिसर $= \left(\frac{-\pi^2}{2}, \ \frac{\pi^2}{2}\right)$ $\quad \left(\because \ \text{range of} \ \sin^{-1}x = \left(\frac{-\pi}{2}, \ \frac{\pi}{2}\right)\right)$

f(x) इसके प्रान्त के अंदर अवकलनीय है और $f'(x) = 0$ का कोई हल नहीं है।
अतः f(x) में कोई क्रांतिक बिन्दु नहीं है।
$f(x) = \pi\sin^{-1}x, \ x \in (-1, 1)$
लेग्रांज प्रमेय के उपयोग से

$$f'(x) = \frac{f(1) - f(-1)}{1 - (-1)}$$

$$\Rightarrow \qquad \frac{\pi}{\sqrt{1-x^2}} = \frac{\pi^2}{2} \qquad\qquad \Rightarrow \qquad x = \pm \frac{\sqrt{\pi^2 - 4}}{\pi} \in (-1, 1)$$

$\therefore$ f(x) के लिए इसके प्रान्त में यहाँ पर दो लेग्रांज अचर है।

Comprehension # 3

If length of perpendicular drawn from points of a curve to a straight line approaches zero along an infinite branch of the curve, the line is said to be an asymptote to the curve. For example, y-axis is an asymptote to $y = \ell nx$ & x-axis is an asymptote to $y = e^{-x}$.

Asymptotes parallel to x-axis :

If $\lim\limits_{x \to \infty} f(x) = e$ (a finite number) then y = e is an asymptote to y = f(x). Similarly if $\lim\limits_{x \to -\infty} f(x) = \alpha$, then $y = \alpha$ is also an asymptote.

Asymptotes parallel to y-axis :

If $\lim\limits_{x \to a} f(x) = \infty$ or $\lim\limits_{x \to a} f(x) = -\infty$, then x = a is an asymptote to y = f(x).

अनुच्छेद # 3

यदि वक्र के बिन्दुओं से सरल रेखा पर लम्ब की लम्बाई शून्य की ओर अग्रसर होती है। जब वक्र की शाखा अनन्त की ओर हो। तब रेखा को वक्र की अनन्तस्पर्शी कहते है। उदाहरण के लिए y-अक्ष, वक्र $y = \ell nx$ की एक अनन्त स्पर्शी है तथा x-अक्ष, वक्र $y = e^{-x}$ की अनन्तस्पर्शी है।

x-अक्ष के समान्तर अनन्तस्पर्शी है।

यदि $\lim\limits_{x \to \infty} f(x) = e$ (एक परिमित संख्या) तब y = e, y = f(x) की अनन्तस्पर्शी है इसी प्रकार यदि $\lim\limits_{x \to -\infty} f(x) = \alpha$, तब $y = \alpha$ भी एक अनन्तस्पर्शी है।

y-अक्ष के समान्तर अनन्तस्पर्शी

यदि $\lim\limits_{x \to a} f(x) = \infty$ या $\lim\limits_{x \to a} f(x) = -\infty$, तब x = a, y = f(x) की अनन्तस्पर्शी है।

7. Number of asymptotes parallel to co-ordinate axes for the function $f(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$ is equal to :

फलन $f(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$ के लिए निर्देशांक अक्षों के समान्तर अनन्तस्पर्शीयों की संख्या बराबर है :

(A) 1 (B) 2 (C*) 3 (D) 4

Sol. $f(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$, $\lim\limits_{x \to \pm\infty} f(x) = 1$ $\Rightarrow$ y = 1 is a horizontal asymbole

Also $\lim\limits_{x \to 1} f(x)$ & $\lim\limits_{x \to 2} f(x)$ approaches ∞ or $-\infty$, hence

x = 1 & x = 2 are vertical asymptotes

Hence number of asymptotes = 3

Hindi. $f(x) = \dfrac{(x+1)(x+2)}{(x-1)(x-2)}$, $\lim\limits_{x \to \pm\infty} f(x) = 1$ $\Rightarrow$ y = 1 क्षैतिज अनन्तस्पर्शी है।

$\lim\limits_{x \to 1} f(x)$ और $\lim\limits_{x \to 2} f(x)$ $x \to \infty$ या $x \to -\infty$ की ओर अग्रसर है।

x = 1 या x = 2 उर्ध्वाधर अनन्तस्पर्शी है।

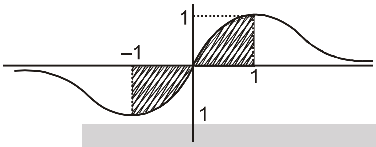
अतः अनन्तस्पर्शियों की संख्या = 3

8. Area bounded by $y = \frac{2x}{x^2+1}$, it's asymptote and ordinates at points of extremum is equal to (in square unit)

वक्र $y = \frac{2x}{x^2+1}$ तथा इसकी अनन्तस्पर्शियों और चरम मानों के बिन्दुओं पर कोटियों से परिबद्ध क्षेत्रफल बराबर है। (वर्ग इकाई में)

(A) ℓn2 (B*) 2ℓn2 (C) ℓn3 (D) 2ℓn3

Sol.

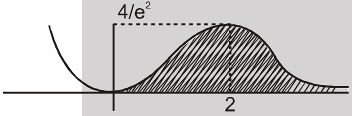


9. Area bounded by $y = x^2e^{-x}$ and it's asymptote in first quadrant is equal to (in square unit)

वक्र $y = x^2e^{-x}$ और इसकी अनन्तस्पर्शियों का प्रथम चतुर्थांश में परिबद्ध क्षेत्र का क्षेत्रफल बराबर है। (वर्ग इकाई में)

(A) 2e (B) e (C) 1 (D*) 2

Sol.



Exercise-3

✍ **Marked questions are recommended for Revision.**
✍ चिन्हित प्रश्न दोहराने योग्य प्रश्न है।

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

भाग - I : JEE (ADVANCED) / IIT-JEE (पिछले वर्षो) के प्रश्न

* **Marked Questions may have more than one correct option.**
* चिन्हित प्रश्न एक से अधिक सही विकल्प वाले प्रश्न है -

1. The value of $\lim\limits_{x\to 0}\dfrac{1}{x^3}\displaystyle\int_0^x \dfrac{t\ell n\ (1+t)}{t^4+4}$ dt is **[IIT-JEE-2010, Paper-1 (3, –1)/84]**

$\lim\limits_{x\to 0}\dfrac{1}{x^3}\displaystyle\int_0^x \dfrac{t\ell n\ (1+t)}{t^4+4}$ dt का मान है– **[IIT-JEE-2010, Paper-1 (3, –1)/84]**

(A) 0 (B*) $\dfrac{1}{12}$ (C) $\dfrac{1}{24}$ (D) $\dfrac{1}{64}$

Sol. $\lim\limits_{x\to 0}\dfrac{x\ell n\ (1+x)}{(x^4+4)\ \times 3x^2} = \lim\limits_{x\to 0}\dfrac{1}{4} \times \dfrac{1}{3} = \dfrac{1}{12}$

2.✍ The value(s) of $\displaystyle\int_0^1 \dfrac{x^4(1-x)^4}{1+x^2}$ dx is (are) **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

$\displaystyle\int_0^1 \dfrac{x^4(1-x)^4}{1+x^2}$ dx का (के) मान है– **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

(A*) $\dfrac{22}{7}-\pi$ (B) $\dfrac{2}{105}$ (C) 0 (D) $\dfrac{71}{15}-\dfrac{3\pi}{2}$

Sol. $\displaystyle\int_0^1 \dfrac{x^4(1-x)^4}{1+x^2}dx = \int_0^1 \dfrac{x^4[(1+x^2)-2x]^2}{1+x^2}\ dx = \int_0^1 \dfrac{x^4[(1+x^2)^2-4x(1+x^2)+4x^2]}{1+x^2}\ dx$

$= \displaystyle\int_0^1 x^4\left[(1+x^2)-4x+\dfrac{4x^2}{1+x^2}\right]\ dx = \int_0^1\left[x^6+x^4-4x^5+\dfrac{4x^6}{1+x^2}\right]\ dx$

Now on polynomial division of x^6 by $1 + x^2$, we obtain

x^6 बहुपद को $1 + x^2$ से विभाजित करने पर

$= \displaystyle\int_0^1\left[x^6+x^4-4x^5+4\left[(x^4-x^2+1)-\dfrac{1}{1+x^2}\right]\right]\ dx = \int_0^1\left[\left(x^6-4x^5+5x^4-4x^2+4\right)-\dfrac{4}{1+x^2}\right]\ dx$

$= \left[\dfrac{x^7}{7}-\dfrac{4x^6}{6}+\dfrac{5\ .\ x^5}{5}-\dfrac{4x^3}{3}+4x\right]_0^1 -4\left[\tan^{-x} x\right]_0^1 = \left(\dfrac{1}{7}-\dfrac{4}{6}+1-\dfrac{4}{3}+4\right)\ -4\ \left(\dfrac{\pi}{4}\right) = \left[\dfrac{1}{7}-\dfrac{12}{6}+5\right]-\pi$

$= \left(\dfrac{1}{7}+3\right)-\pi = \dfrac{22}{7}-\pi$

3*.✍ Let f be a real-valued function defined on the interval (0, ∞) by $f(x) = \ell n\, x + \int_0^x \sqrt{1+\sin t}\, dt$. Then which of the following statement(s) is (are) true? **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B*) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C*) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)| + |f'(x)| \le \beta$ for all $x \in (0, \infty)$

माना कि वास्तविक मानों वाला फलन f अन्तराल (0, ∞) पर $f(x) = \ell n\, x + \int_0^x \sqrt{1+\sin t}\, dt$ द्वारा परिभाषित है। तो निम्न में से कौनसे वक्तव्य सत्य हैं? **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

(A) सभी $x \in (0, \infty)$ के लिए $f''(x)$ का अस्तित्व है।

(B) सभी $x \in (0, \infty)$ के लिए $f'(x)$ का अस्तित्व है, $(0, \infty)$ पर f' सतत् है परन्तु $(0, \infty)$ पर f' अवकलनीय नहीं हैं।

(C) ऐसे $\alpha > 1$ का अस्तित्व है कि सभी $x \in (\alpha, \infty)$ के लिए $|f'(x)| < |f(x)|$ हो।

(D) ऐसे $\beta > 0$ का अस्तित्व है कि सभी $x \in (0, \infty)$ के लिए $|f(x)| + |f'(x)| \le \beta$ हो।

Sol. $f(x) = \ell n\, x + \int_0^x \sqrt{1+\sin t}\, dt$

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$$

$$f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$$

(A) f'' is not defined for $x = -\frac{\pi}{2} + 2n\pi$, $n \in I$

so (A) is wrong

(B) $f'(x)$ always exist for $x > 0$

(C) $| f' | < | f |$

Since $f' > 0$ and $f > 0$

$f' < f$

$$\frac{1}{x} + \sqrt{1+\sin x} < \ell n\, x + \int_0^x \sqrt{1+\sin x}\, dx$$

LHS is bounded RHS is Increasing with range ∞

So there exist some α beyond which RHS is greater than LHS

(D) $| f | + | f' | \le \beta$ is wrong as f is MI & its range is not bounded, while β is finite

Hindi $f(x) = \ell n\, x + \int_0^x \sqrt{1+\sin t}\, dt$

$$f'(x) = \frac{1}{x} + \sqrt{1+\sin x}$$

$$f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1+\sin x}}$$

(A) $x = -\frac{\pi}{2} + 2n\pi$, $n \in I$ के लिए f'' अपरिभाषित है अतः (A) गलत है।

(B) $x > 0$ के लिए $f'(x)$ सदैव विद्यमान है।

(C) $| f' | < | f |$

चूँकि $f' > 0$ और $f > 0$

$f' < f$

$$\frac{1}{x} + \sqrt{1+\sin x} < \ell n\, x + \int_0^x \sqrt{1+\sin x}\, dx$$

LHS परिबद्ध है RHS वर्धमान है जिसका परिसर ∞ है।

इसलिए किसी α के मान के लिए RHS, LHS से बड़ा है।

(D) $|f| + |f'| \le \beta$ यह गलत है क्योंकि f एकदिष्ट वर्धमान है और इसका परिसर परिबद्ध नहीं है जबकि β परिमित है।

4.✍ For any real number, let [x] denote the largest integer less than or equal to x. Let f be a real valued function defined on the interval [–10, 10] by **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

$$f(x) = \begin{cases} x-[x] & \text{if } [x] \text{ is odd,} \\ 1+[x]-x & \text{if } [x] \text{ is even} \end{cases}$$

Then the value of $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$ is

दिया है कि किसी वास्तविक संख्या x के लिए [x], अधिकतम पूर्णांक ≤ x को दर्शाता है। यदि अन्तराल [–10, 10] पर वास्तविक मानों वाला फलन f निम्न प्रकार से परिभाषित है **[IIT-JEE-2010, Paper-1 (3, 0)/84]**

$$f(x) = \begin{cases} x-[x] & \text{यदि } [x] \text{ विषम है।} \\ 1+[x]-x & \text{यदि } [x] \text{ सम है।} \end{cases}$$

तो $\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos \pi x\, dx$ का मान है।

Ans. 4

Sol. $f(x) = \begin{cases} \{x\} & , \ 2n-1 \le x < 2n \\ 1-\{x\} & , \ 2n \le x < 2n+1 \end{cases}$

Clearly f(x) is a periodic function with period = 2

Hence f(x) . cos πx is also periodic with period = 2

$$\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos(\pi x)\ dx = \pi^2 \int_0^2 f(x)\ \cos(\pi x)\ dx = \pi^2 \int_0^1 ((1-\{x\})+\{-x\})\cos(\pi x)\ dx$$

$$= \quad 2\pi^2 \int_0^1 (-x\cos \pi x)\ dx = -2\pi^2 \left[\frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1 = -2\pi^2\left(-\frac{2}{\pi^2}\right) = 4$$

Hindi $f(x) = \begin{cases} \{x\} & , \ 2n-1 \le x < 2n \\ 1-\{x\} & , \ 2n \le x < 2n+1 \end{cases}$

स्पष्टतया f(x) आवर्ती फलन है जिसका आवर्तकाल = 2

अतः f(x) . cos πx भी आवर्ती फलन है जिसका आवर्तकाल = 2

$$\frac{\pi^2}{10}\int_{-10}^{10} f(x)\cos(\pi x)\ dx = \pi^2 \int_0^2 f(x)\ \cos(\pi x)\ dx = \pi^2 \int_0^1 ((1-\{x\})+\{-x\})\cos(\pi x)\ dx$$

$$= \quad 2\pi^2 \int_0^1 (-x\cos \pi x)\ dx = -2\pi^2 \left[\frac{x\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2}\right]_0^1 = -2\pi^2\left(-\frac{2}{\pi^2}\right) = 4$$

5.✍ Let f be a real-valued function defined on the interval (–1, 1) such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\ dt$, for all $x \in (-1, 1)$ and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to

[IIT-JEE-2010, Paper-2 (5, –2)/84]

माना कि अन्तराल (–1, 1) पर वास्तविक मानों वाला फलन f इस प्रकार परिभाषित है कि सभी $x \in (-1, 1)$ के लिए $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4+1}\ dt$ तथा f^{-1} फलन f का प्रतिलोम (inverse) है। तो $(f^{-1})'(2)$ का मान है–

[IIT-JEE-2010, Paper-2 (5, –2)/84]

(A) 1 (B*) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{e}$

Sol. $f(x) = e^x\left(2+\int_0^x \sqrt{t^4+1}\ dt\right)$

Let माना $g(x) = f^{-1}(x)$ $\Rightarrow$ $g(f(x)) = x$

$\Rightarrow$ $g'(f(x))\, f'(x) = 1$

$\Rightarrow$ $g'(2) = \dfrac{1}{f'(0)}$ $(\because f(0) = 2)$

Now अब $f'(x) = e^x\left(2+\int_0^x \sqrt{t^4+1}\ dt\right) + e^x\sqrt{x^4+1}$ (Applying Leibinitz Rule) (लेबनीज नियम लगाने पर)

$\Rightarrow$ $f'(0) = 2 + 1 = 3$

$\Rightarrow$ $g'(2) = \dfrac{1}{3}$

$\Rightarrow$ $(f^{-1})'(2) = \dfrac{1}{3}$

Comprehension (6 to 8)

Consider the polynomial

$f(x) = 1 + 2x + 3x^2 + 4x^3$

Let s be the sum of all distinct real roots of f(x) and let t = |s|

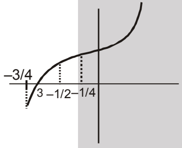
6. The real number s lies in the interval. **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) $\left(-\dfrac{1}{4},\ 0\right)$ (B) $\left(-11,\ -\dfrac{3}{4}\right)$ (C*) $\left(-\dfrac{3}{4},\ -\dfrac{1}{2}\right)$ (D) $\left(0\ ,\ \dfrac{1}{4}\right)$

Sol. $f(x) = 1 + 2x + 3x^2 + 4x^3$

$f'(x) = 2 + 6x + 12x^2 > 0$ [as a > 0, D < 0]

f(x) is increasing function so it can utmost one real root.



Using inter mediate value theorem

$f\left(-\dfrac{3}{4}\right) . f\left(-\dfrac{1}{2}\right) < 0$

7. The area bounded by the curve y = f(x) and the lines x = 0, y = 0 and x = t, lies in the interval **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A*) $\left(\dfrac{3}{4}\ ,\ 3\right)$ (B) $\left(\dfrac{21}{64}\ ,\ \dfrac{11}{16}\right)$ (C) (9, 10) (D) $\left(0\ ,\ \dfrac{21}{64}\right)$

Sol. By estimation of integration

$\int_0^{1/2} f(x)\ dx < \int_0^{t} f(x)\ dx < \int_0^{3/4} f(x)\ dx$

$\Rightarrow \dfrac{15}{16} < \int_0^{t} f(x)\ dx < \dfrac{525}{256}$

Hence option (A) is correct

8. The function f′(x) is **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) increasing in $\left(-t\ ,\ \dfrac{1}{4}\right)$ and decreasing in $\left(-\dfrac{1}{4}\ ,\ t\right)$

(B*) decreasing in $\left(-t\ ,\ -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4},\ t\right)$

(C) increasing in (–t, t)

(D) decreasing in (–t, t)

Sol. $f'(x) = 2 + 6x + 12x^2$

$\Rightarrow f''(x) = 6 + 24x$

$\Rightarrow f''(x) = 6\ (4x + 1) > 0 \Rightarrow x > -\frac{1}{4}$

अनुच्छेद

दिया है बहुपद $f(x) = 1 + 2x + 3x^2 + 4x^3$

मानाकि f(x) के सभी भिन्न, वास्तविक मूलों का योग s है तथा t = |s| है।

6. वास्तविक संख्या s , निम्न अन्तराल में है– **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

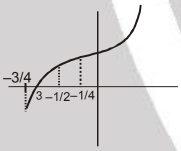
(A) $\left(-\frac{1}{4},\ 0\right)$ (B) $\left(-11,\ -\frac{3}{4}\right)$ (C*) $\left(-\frac{3}{4}\ ,\ -\frac{1}{2}\right)$ (D) $\left(0\ ,\ \frac{1}{4}\right)$

Sol. $f(x) = 1 + 2x + 3x^2 + 4x^3$

$f'(x) = 2 + 6x + 12x^2 > 0$ [जैसे कि a > 0, D < 0]

f(x) वर्धमान फलन है इसलिए इसका अधिक से अधिक एक वास्तविक मूल हो सकता है

माध्यमान प्रमेय की सहायता से



$f\left(-\frac{3}{4}\right) . f\left(-\frac{1}{2}\right) < 0$

7. वक्र y = f(x) तथा सरल रेखाओं x = 0, y = 0 एवं x = t से परिबद्ध क्षेत्र का क्षेत्रफल निम्न अन्तराल में है– **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A*) $\left(\frac{3}{4}\ ,\ 3\right)$ (B) $\left(\frac{21}{64}\ ,\ \frac{11}{16}\right)$ (C) (9, 10) (D) $\left(0\ ,\ \frac{21}{64}\right)$

Sol. समाकलन से (estimation)

$$\int_0^{1/2} f(x)\ dx < \int_0^{t} f(x)\ dx < \int_0^{3/4} f(x)\ dx$$

$$\Rightarrow \frac{15}{16} < \int_0^{t} f(x)\ dx < \frac{525}{256}$$

इस प्रकार विकल्प (A) सही है।

8. फलन f′(x) है– **[IIT-JEE 2010, Paper-2, (3, –1), 79]**

(A) $\left(-t\ ,\ \frac{1}{4}\right)$ में वर्धमान (increasing) तथा $\left(-\frac{1}{4}\ ,\ t\right)$ में ह्रासमान (decreasing) है।

(B*) $\left(-t\ ,\ -\frac{1}{4}\right)$ में ह्रासमान तथा $\left(-\frac{1}{4},\ t\right)$ में वर्धमान है।

(C) (–t, t) में वर्धमान है।

(D) (–t, t) में ह्रासमान है।

Sol. $f'(x) = 2 + 6x + 12x^2$

$\Rightarrow f''(x) = 6 + 24x$

$\Rightarrow f''(x) = 6\ (4x + 1) > 0 \Rightarrow x > -\frac{1}{4}$

9. The value of $\int_{\sqrt{\ell n2}}^{\sqrt{\ell n3}} \frac{x\sin x^2}{\sin x^2+\sin(\ell n6-x^2)}$ dx is

$\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x\sin x^2}{\sin x^2+\sin(\ln 6-x^2)}$ dx का मान है– **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A*) $\frac{1}{4}\ \ell n\ \frac{3}{2}$ (B) $\frac{1}{2}\ \ell n\ \frac{3}{2}$ (C) $\ell n\ \frac{3}{2}$ (D) $\frac{1}{6}\ \ell n\ \frac{3}{2}$

Ans. (A)

Sol. Put $x^2 = t$

$x\,dx = \frac{dt}{2}$

$I = \int_{\ell n2}^{\ell n3} \frac{\sin t}{\sin t+\sin\ (\ell n6-t)}\ .\ \frac{dt}{2}$ ......(1)

apply $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

$I = \frac{1}{2}\int_{\ell n2}^{\ell n3} \frac{\sin(\ell n6-t)}{\sin(\ell n6-t)+\sin t}$ dt ....(2)

adding (1) and (2)

$2I = \frac{1}{2}\int_{\ell n2}^{\ell n3} 1.dt$

$\Rightarrow I = \frac{1}{4}\ell n\frac{3}{2}$

Hindi $x^2 = t$ रखने पर

$x\,dx = \frac{dt}{2}$

$I = \int_{\ell n2}^{\ell n3} \frac{\sin t}{\sin t+\sin\ (\ell n6-t)}\ .\ \frac{dt}{2}$ ......(1)

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ गुणधर्म से

$I = \frac{1}{2}\int_{\ell n2}^{\ell n3} \frac{\sin(\ell n6-t)}{\sin(\ell n6-t)+\sin t}$ dt ....(2)

(1) और (2) को जोड़ने पर

$2I = \frac{1}{2}\int_{\ell n2}^{\ell n3} 1.dt$

$\Rightarrow I = \frac{1}{4}\ell n\frac{3}{2}$

10. Let the straight line x = b divide the area enclosed by $y = (1-x)^2$, y = 0, and x = 0 into two parts R_1 ($0 \le x \le b$) and R_2($b \le x \le 1$) such that $R_1 - R_2 = \frac{1}{4}$. Then b equals **[Area between curve]**

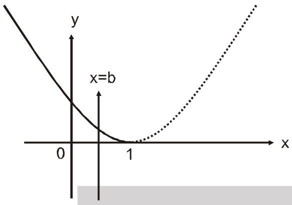
यदि $y = (1-x)^2$, y = 0 और x = 0 द्वारा परिबद्ध क्षेत्र को सरल रेखा x = b दो हिस्सों, R_1 ($0 \le x \le b$) और R_2($b \le x \le 1$) में इस प्रकार विभाजित करती है कि $R_1 - R_2 = \frac{1}{4}$, तो b का मान है– **[IIT-JEE 2011, Paper-1, (3, –1), 80]**

(A) $\frac{3}{4}$ (B*) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

Ans. (B)

Sol. $R_1 = \int_0^b (x-1)^2 dx = \left.\frac{(x-1)^3}{3}\right|_0^b = \frac{(b-1)^3+1}{3}$

also $R_2 = \int_b^1 (x-1)^2 dx = \left.\frac{(x-1)^3}{3}\right|_b^1 = -\frac{(b-1)^3}{3}$



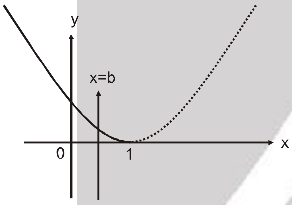
$\Rightarrow \quad R_1 - R_2 = \frac{2(b-1)^3}{3} + \frac{1}{3}$

$\Rightarrow \quad \frac{1}{4} = \frac{2(b-1)^3}{3} + \frac{1}{3} \qquad \Rightarrow \quad (b-1)^3 = -\frac{1}{8}$

$\Rightarrow \quad b = \frac{1}{2}$

Hindi $R_1 = \int_0^b (x-1)^2 dx = \left.\frac{(x-1)^3}{3}\right|_0^b = \frac{(b-1)^3+1}{3}$

तथा $R_2 = \int_b^1 (x-1)^2 dx = \left.\frac{(x-1)^3}{3}\right|_b^1 = -\frac{(b-1)^3}{3}$



$\Rightarrow \quad R_1 - R_2 = \frac{2(b-1)^3}{3} + \frac{1}{3}$

$\Rightarrow \quad \frac{1}{4} = \frac{2(b-1)^3}{3} + \frac{1}{3} \qquad \Rightarrow \quad (b-1)^3 = -\frac{1}{8}$

$\Rightarrow \quad b = \frac{1}{2}$

11. Let $f : [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} x \; f(x) \; dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$, and the x-axis. Then

मान लीजिए कि $f : [-1, 2] \to [0, \infty)$ एक ऐसा सतत फलन है जो कि अन्तराल $[-1, 2]$ में x के सभी मानों के लिये $f(x) = f(1-x)$ को संतुष्ट करता है। यदि $R_1 = \int_{-1}^{2} x \; f(x) \; dx$, है और R_2 उस क्षेत्र का क्षेत्रफल है जो $y = f(x)$, $x = -1$,

$x = 2$, तथा x-अक्ष द्वारा परिबद्ध है, तब **[IIT-JEE 2011, Paper-2, (3, –1), 80]**

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C*) $2R_1 = R_2$ (D) $3R_1 = R_2$

[Area between curve]

Ans. (C)

Sol. $R_2 = \int_{-1}^{2} f(x) \; dx$ and $R_1 = \int_{-1}^{2} x f(x) \; dx$

$\Rightarrow \quad = \int_{-1}^{2}(1-x)f(1-x)\ \ dx$

$= \int_{-1}^{2}(1-x)f(x)\ \ dx$

$R_1 = R_2 - R_1$

$\Rightarrow \quad 2R_1 = R_2$

Hindi $R_2 = \int_{-1}^{2} f(x)\ \ dx$ तथा $R_1 = \int_{-1}^{2} xf(x)\ \ dx$

$\Rightarrow \quad = \int_{-1}^{2}(1-x)f(1-x)\ \ dx$

$= \int_{-1}^{2}(1-x)f(x)\ \ dx$

$R_1 = R_2 - R_1$

$\Rightarrow \quad 2R_1 = R_2$

12.* If S be the area of the region enclosed by $y = e^{-x^2}$, y = 0, x = 0, and x = 1. Then **[Area]**

यदि $y = e^{-x^2}$, y = 0, x = 0 और x = 1 द्वारा परिबद्ध (enclosed) क्षेत्र का क्षेत्रफल S है तो

[IIT-JEE 2012, Paper-1, (4, 0), 70]

(A*) $S \geq \frac{1}{e}$ (B*) $S \geq 1-\frac{1}{e}$ (C) $S \leq \frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)$ (D*) $S \leq \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{2}}\right)$

Sol. Ans (ABD)

$I = \int_{0}^{1} e^{-x^2} dx$

$-x^2 \leq 0$

$e^{-x^2} \leq 1$

$\int_{0}^{1} e^{-x^2} dx \leq 1$

$x^2 \leq x \Rightarrow -x^2 \geq -x \Rightarrow e^{-x^2} \geq e^{-x}$

$\Rightarrow I \geq \int_{0}^{1} e^{-x} dx$

$\geq -\left(e^{-x}\right)_0^1$

$\geq -\left(\frac{1}{e}-1\right)$

$I \geq 1 - \frac{1}{e}$ ⇒ (B) is correct

Since If $I \geq 1- \frac{1}{e} \Rightarrow I > \frac{1}{e}$ ⇒ (A) is correct

$I < \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{e}} \times (1 - \frac{1}{\sqrt{2}})$

So Ans. D

[Hence Answers are A, B and D]

Hindi Ans (ABD)

$I = \int_{0}^{1} e^{-x^2} dx$

$-x^2 \le 0$

$e^{-x^2} \le 1$

$\int_0^1 e^{-x^2} dx \le 1$

$x^2 \le x \Rightarrow -x^2 \ge -x \Rightarrow e^{-x^2} \ge e^{-x}$

$\Rightarrow I \ge \int_0^1 e^{-x} dx$

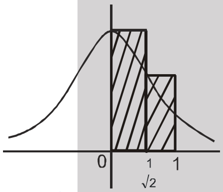
$\ge -\left(e^{-x}\right)_0^1$

$\ge -\left(\frac{1}{e}-1\right)$

$I \ge 1-\frac{1}{e}$ $\Rightarrow$ विकल्प (B) सही है

चूंकि $I \ge 1-\frac{1}{e} \Rightarrow I > \frac{1}{e}$ $\Rightarrow$ विकल्प (A) सही है

$I < \frac{1}{\sqrt{2}} \times 1 + \frac{1}{\sqrt{e}} \times \left(1-\frac{1}{\sqrt{2}}\right)$



अतः विकल्प (D) सही है

[अतः विकल्पA, B तथा D सही है]

13. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ell n\frac{\pi+x}{\pi-x}\right)\cos x \ dx$ is **(Definite Integration)**

समाकल $\int_{-\pi/2}^{\pi/2}\left(x^2+\ell n\frac{\pi+x}{\pi-x}\right)\cos x \ dx$ का मान निम्न है– **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) 0 (B*) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$

Sol. Ans. (B)

$$\int_{-\pi/2}^{\pi/2}\left(x^2+\ell n\left(\frac{\pi+x}{\pi-x}\right)\right)\cos x \, dx = 2\int_0^{\pi/2} x^2\cos x dx + 0 \qquad \left(\because \ell n\left(\frac{\pi+x}{\pi-x}\right) \text{is an odd function}\right)$$

$$= 2\left[\left(x^2\sin x\right)_0^{\pi/2} - \int_0^{\pi/2} 2x\sin x \ dx\right]$$

$$= 2\left(\frac{\pi^2}{4}-0\right) - 4\int_0^{\pi/2} x\sin x \ dx$$

$$= \frac{\pi^2}{2} - 4\left[\left(-x\cos x\right)_0^{\pi/2} + \int_0^{\pi/2}\cos x \ dx\right]$$

$$= \frac{\pi^2}{2} - 4$$

Hindi Ans. (B)

$$\int_{-\pi/2}^{\pi/2}\left(x^2+\ell n\left(\frac{\pi+x}{\pi-x}\right)\right)\cos x \, dx = 2\int_0^{\pi/2} x^2\cos x dx + 0 \qquad \left(\because \ell n\left(\frac{\pi+x}{\pi-x}\right) \text{एक विषम फलन है।}\right)$$

$$= 2\left[\left(x^2 \sin x\right)_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x \ \ dx\right]$$

$$= 2\left(\frac{\pi^2}{4} - 0\right) - 4\int_0^{\pi/2} x \sin x \ \ dx$$

$$= \frac{\pi^2}{2} - 4 \ \left[\left(-x \cos x\right)_0^{\pi/2} + \int_0^{\pi/2} \cos x \ \ dx\right]$$

$$= \frac{\pi^2}{2} - 4$$

$$= \frac{\pi^2}{2} - 4 \ \left[\left(-x \cos x\right)_0^{\pi/2} + \int_0^{\pi/2} \cos x \ \ dx\right]$$

$$= \frac{\pi^2}{2} - 4$$

Paragraph for Question Nos. 14 to 15

प्रश्न 14 से 15 के लिए अनुच्छेद

Let f(x) = $(1 - x)^2 \sin^2 x + x^2$ for all $x \in$ IR and let g(x) = $\int_1^x \left(\frac{2(t-1)}{t+1} - \ell nt\right)$ f(t) dt for all $x \in (1, \infty)$.

माना कि f(x) = $(1 - x)^2 \sin^2 x + x^2$ जहाँ $x \in$ IR और g(x) = $\int_1^x \left(\frac{2(t-1)}{t+1} - \ell nt\right)$ f(t) dt, जहाँ $x \in (1, \infty)$.

14. Which of the following is true ? **[IIT-JEE 2012, Paper-2, (3, –1), 66]**

(A) g is increasing on (1, ∞)

(B*) g is decreasing on (1, ∞)

(C) g is increasing on (1, 2) and decreasing on (2, ∞)

(D) g is decreasing on (1, 2) and increasing on (2, ∞)

निम्न में से कौन सा कथन सही है ?

(A) (1, ∞) में g वर्धमान (increasing) है।

(B*) (1, ∞) में g ह्यसमान (decreasing) है।

(C) (1, 2) में g वर्धमान (increasing) है और (2, ∞) में ह्यसमान (decreasing) है।

(D) (1, 2) में g ह्यसमान (decreasing) है और (2, ∞) में वर्धमान (increasing) है।

Sol. Ans. (B)

f(x) = $(1 - x)^2 \sin^2 x + x^2 : x \in R$

$$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)dt$$

$$\therefore \ g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right) \ f(x) \ . \ 1$$

$$\text{let } \phi(x) = \frac{2(x-1)}{x+1} - \ln x$$

$$\phi'(x) = \frac{2[(x+1)-(x-1).1]}{(x+1)^2} - \frac{1}{x}$$

$$= \frac{4}{(x+1)^2} - \frac{1}{x}$$

$$= \frac{-x^2+2x-1}{x(x+1)^2} = \frac{-(x-1)^2}{x(x+1)^2}$$

$\therefore$ $\phi'(x) \le 0$

$\therefore$ for $x \in (1, \infty)$, $\phi(x) < 0$

$\therefore$ $g'(x) < 0$ for $x \in (1, \infty)$

Hindi Ans. (B)

$f(x) = (1 - x)^2 \sin^2 x + x^2 : x \in R$

$g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t)dt$

$\therefore\ g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right) f(x) . 1$

let $\phi(x) = \frac{2(x-1)}{x+1} - \ln x$

$\phi'(x) = \frac{2[(x+1)-(x-1).1]}{(x+1)^2} - \frac{1}{x}$

$= \frac{4}{(x+1)^2} - \frac{1}{x}$

$= \frac{-x^2+2x-1}{x(x+1)^2} = \frac{-(x-1)^2}{x(x+1)^2}$

$\therefore$ $\phi'(x) \le 0$

$\therefore$ for $x \in (1, \infty)$, $\phi(x) < 0$

$\therefore$ $g'(x) < 0$ $x \in (1, \infty)$ के लिए

15. Consider the statements :

P : There exists some $x \in$ IR such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in$ IR such that $2f(x) + 1 = 2x(1 + x)$

Then

(A) both P and Q are true (B) P is true and Q is false

(C*) P is false and Q is true (D) both P and Q are false

दिये गये कथन है :

P : एक ऐसी संख्या $x \in$ IR का अस्तित्व है जिसके लिए $f(x) + 2x = 2(1 + x^2)$

Q : एक ऐसी संख्या $x \in$ IR का अस्तित्व है जिसके लिए $2f(x) + 1 = 2x(1 + x)$

तब निम्न में से कौनसा कथन सही है ?

(A) P और Q दोनों सत्य है। (B) P सत्य है और Q असत्य है।

(C*) P असत्य है और Q सत्य है। (D) P और Q दोनों असत्य है।

Sol. **Ans. (C)**

$f(x) + 2x = (1 - x)^2 \sin^2 x + x^2 + 2x$

$\because$ $f(x) + 2x = 2(1 + x^2)$

$\Rightarrow$ $(1 - x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$

$(1 - x)^2 \sin^2 x = x^2 - 2x + 1 + 1$

$= (1 - x)^2 + 1$

$\Rightarrow$ $(1 - x)^2 \cos^2 x = -1$

which can never be possible

P is not true

$\Rightarrow$ Let $H(x) = 2f(x) + 1 - 2x(1 + x)$

$H(0) = 2f(0) + 1 - 0 = 1$

$H(1) = 2f(1) + 1 - 4 = -3$

$\Rightarrow$ so H(x) has a solution

so Q is true

Hindi. $f(x) + 2x = (1 - x)^2 \sin^2 x + x^2 + 2x$

$\because$ $f(x) + 2x = 2(1 + x^2)$

$\Rightarrow$ $(1 - x)^2 \sin^2 x + x^2 + 2x = 2 + 2x^2$

$(1 - x)^2 \sin^2 x = x^2 - 2x + 1 + 1$

$= (1 - x)^2 + 1$

$\Rightarrow$ $(1 - x)^2 \cos^2 x = -3$

जो कि सम्भव नही हो सकता

P असत्य है।

⇒ मानाकि H(x) = 2f(x) + 1 – 2x(1 + x)

H(0) = 2f(0) + 1 – 0 = 1

H(1) = 2f(1) + 1 – 4 = – 1

⇒ इसलिए H(x) का एक हल है।

अतः Q सत्य है।

16.* If $f(x) = \int_0^x e^{t^2}(t-2)\ \ (t-3)\ \ dt$ for all x ∈ (0, ∞), then **(Definite Integration)**

[IIT-JEE 2012, Paper-2, (4, 0), 66]

(A*) f has a local maximum at x = 2

(B*) f is decreasing on (2, 3)

(C*) there exists some c ∈ (0, ∞) such that f′′(c) = 0

(D*) f has a local minimum at x = 3

यदि सभी x ∈ (0, ∞) के लिये $f(x) = \int_0^x e^{t^2}(t-2)\ \ (t-3)\ \ dt$, तब

(A*) x = 2 पर f का स्थानीय उच्चतम (local maximum) है।

(B*) (2, 3) में f ह्रासमान (decreasing) है।

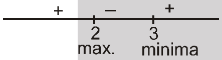
(C*) किसी संख्या c ∈ (0, ∞) के लिये f′′(c) = 0 है।

(D*) x = 3 पर f का स्थानीय न्यूनतम (local minimum) है।

Sol. Ans. (ABCD)

$f(x) = \int_0^x e^{t^2}\ \cdot\ (t-2)(t-3)\ \ dt$

$f'(x) = 1\ \cdot\ e^{x^2}\ \cdot\ (x-2)\ (x-3)$



(i) x = 2 is local maxima (ii) x = 3 is local minima (iii) It is decreasing in x ∈ (2, 3)

(i) x = 2 पर स्थानीय अधिकतम है (ii) x = 3 पर स्थानीय न्यूनतम है (iii) यह x ∈ (2, 3) के ह्रासमान है

(iv) $f''(x) = e^{x^2}\ \cdot (x-2) + e^{x^2}\ \ (x-3) + 2x\ e^{x^2}\ \ (x-2)\ (x-3)$

$= \cdot\ [x - 2 + x - 3 + 2x(x-2)(x-3)]$

$f''(x) = 0$

$f''(x) = e^{x^2}\ \ (2x^3 - 10x^2 + 14x - 5)$

f′′(0) < 0 and f′′(1) > 0

so f′′(c) = 0 where c ∈ (0, 1)

इसीलिए f′′(c) = 0 जहाँ c ∈ (0, 1)

17. The area enclosed by the curves y = sinx + cosx and y = |cosx – sinx| over the interval $\left[0,\ \frac{\pi}{2}\right]$ is

अन्तराल $\left[0,\ \frac{\pi}{2}\right]$ पर वक्रों y = sinx + cosx तथा y = |cosx – sinx| द्वारा परिबद्ध क्षेत्रफल है– **(Area under curve) XII**

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

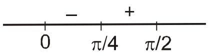
(A) $4\ \left(\sqrt{2}-1\right)$ (B*) $2\sqrt{2}\ \left(\sqrt{2}-1\right)$ (C) $2\ \left(\sqrt{2}+1\right)$ (D) $2\sqrt{2}\ \left(\sqrt{2}+1\right)$

Sol. (B)

Given दिया है y = sin x + cos x x ∈ [0, π/2]

$\frac{dy}{dx} = \cos x - \sin x$

$$y = |\cos x - \sin x| = \begin{bmatrix} \cos\ x - \sin\ x & x \in [0, \pi/4] \\ \sin x - \cos x & x \in [\pi/4\ ,\ \pi/2] \end{bmatrix}$$



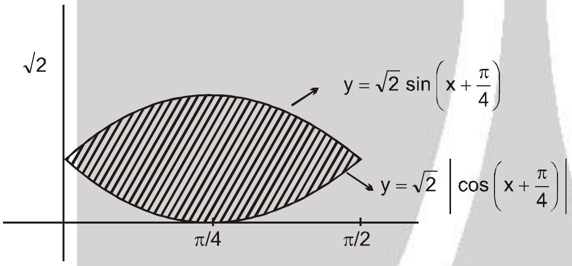
required area अभीष्ट क्षेत्रफल = $\int_0^{\pi/4} |(\sin x + \cos x) - (\cos x - \sin x)| dx + \int_{\pi/4}^{\pi/2} |2\cos x|\ dx$

$= \int_0^{\pi/4} |2\sin x|\ dx + \int_{\pi/4}^{\pi/2} |2\cos x|\,dx\ =\ 2\ \left(-\cos x\right)_0^{\pi/4} + 2\left(\sin x\right)_{\pi/4}^{\pi/2} = 2\left[-\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}\right]$

$= 2\left(2 - \frac{2}{\sqrt{2}}\right)$

$= 2\ \left(2 - \sqrt{2}\right)$

$= 4 - 2\sqrt{2}$



$= 2\sqrt{2}\ \left(\sqrt{2} - 1\right)$

18. Let f : $\left[\frac{1}{2}, 1\right]$ → R (the set of all real numbers) be a positive, non-constant and differentiable function such that f′(x) < 2 f(x) and $f\left(\frac{1}{2}\right)$ = 1. Then the value of $\int_{1/2}^{1} f(x)\ \ dx$ lies in the interval

(Definite Integration) XII

माना कि f : $\left[\frac{1}{2}, 1\right]$ → R (सभी वास्तविक संख्याओं का समुच्चय) एक धनात्मक, अचरेतर तथा अवकलनीय फलन है जिसके लिए f′(x) < 2 f(x) तथा $f\left(\frac{1}{2}\right) = 1$ है, तब $\int_{1/2}^{1} f(x)\ \ dx$ का मान निम्न अन्तराल में है–

[JEE (Advanced) 2013, Paper-1, (2, 0)/60]

(A) (2e – 1, 2e)
(B) (e – 1, 2e – 1)
(C) $\left(\frac{e-1}{2},\ e-1\right)$
(D*) $\left(0\ ,\ \frac{e-1}{2}\right)$

Sol. **(D)**

f′(x) – 2f(x) < 0

$\frac{d}{dx}\ (e^{-2x} f(x)) < 0$

⇒ e^{-2x} f(x) is decreasing ह्रासमान है

⇒ x > 1/2

$e^{-2x} f(x) < 1/e$

⇒ $f(x) < e^{2x-1}$

$\Rightarrow 0 < \int_{1/2}^{1} f(x)\ \ dx < \int_{1/2}^{1} \left(e^{2x-1}\right)\ \ dx \qquad \Rightarrow \qquad 0 < \int_{1/2}^{1} f(x)\ \ dx < \frac{e-1}{2}$

19*. For a ∈ R (the set of all real numbers), a ≠ –1, $\lim_{n\to\infty}\frac{(1^a+2^a+....+n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+...+(na+n)]}=\frac{1}{60}$. Then a =

a ∈ R (सभी वास्तविक संख्याओं का समुच्चय), a ≠ –1 $\lim_{n\to\infty}\frac{(1^a+2^a+....+n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+...+(na+n)]}=\frac{1}{60}$ के लिए तब a= **[JEE (Advanced) 2013, Paper-2, (3, –1)/60]** **(Limit)**

(A) 5 (B*) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$;

Sol. **(B) (JEE given B, D answer)**

$$\frac{2\sum r^a}{(n+1)^{a-1}\left(2n^2a+n^2+n\right)} \Rightarrow \frac{2\sum_{r=1}^{n}\left(\frac{r}{n}\right)^a}{(1+1/n)^{a-1}\left(2n^2a+n^2+n\right)} \Rightarrow \frac{2\int_0^1 x^a dx}{2a+1}$$

$$\frac{2}{(2a+1)(a+1)}=\frac{1}{60}$$

120 = (2a + 1) (a + 1)

a = 7,–17/2 **(–17/2 reject)**

20.* Let f:[a, b] → [1, ∞) be a continuous function and let g : R → R be defined as

$$g(x) = \begin{cases} 0 & \text{if} \quad x < a, \\ \int_a^x f(t)dt & \text{if} \quad a \le x \le b, \\ \int_a^b f(t)dt & \text{if} \quad x > b. \end{cases}$$, Then **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

(A*) g(x) is continuous but not differentiable at a

(B) g (x) is differentiable on R

(C*) g(x) is continuous but not differentiable at b

(D) g(x) is continuous and differentiable at either a or b but not both

माना कि f:[a, b] → [1, ∞) एक संतत फलन है तथा g : R → R निम्नानुसार

$$g(x) = \begin{cases} 0 & \text{यदि} \quad x < a, \\ \int_a^x f(t)dt & \text{यदि} \quad a \le x \le b, \\ \int_a^b f(t)dt & \text{यदि} \quad x > b. \end{cases}$$ परिभाषित है, तब

(A) a पर g(x) संतत (continuous) है परन्तु अवकलनीय (differentiable) नहीं है।

(B) R पर g(x) अवकलनीय है।

(C) b पर g(x) संतत है परन्तु अवकलनीय नहीं है।

(D) a या b पर g(x) संतत एवं अवकलनीय है परन्तु दोनों पर नही।

Ans. **(AC)**

Sol. It may be discontinuous at x = a or x = b

$\lim_{x\to a^-} g(x) = 0$

$$\lim_{x\to a^+} g(x) = \lim_{x\to a^+}\int_a^x f(t)dt = \int_a^a f(t)dt = 0$$

$g(a) = \int_{a}^{a} f(t)dt \;\; = 0$

Similarly at x = b we will get continuous

So g(x) is continuous $\forall$ x $\in$ R

$$g'(x) = \left[\begin{array}{ll} 0 & x < a \\ f(x) & a \le x \le b \\ 0 & x > b \end{array}\right.$$

$g'(a^-) = 0 \qquad g'(b^-) = f(b)$

$g'(a^+) = f(a) \qquad g'(b^+) = 0$

Since f(x) co-domain is [1, ∞) f(a) & f(b) can never be zero.

Hence it is non derivable at x = a & x = b.

Hindi. यह x = a या x = b पर यह असतत् हो सकता है।

$\lim\limits_{x\to a^-} g(x) = 0$

$\lim\limits_{x\to a^+} g(x) = \lim\limits_{x\to a^+} \int_{a}^{x} f(t)dt \;\; = \int_{a}^{a} f(t)dt \;\; = 0$

$g(a) = \int_{a}^{a} f(t)dt \;\; = 0$

इसी प्रकार x = b पर यह सतत् होगा

अतः g(x), $\forall$ x $\in$ R पर सतत् है

$$g'(x) = \left[\begin{array}{ll} 0 & x < a \\ f(x) & a \le x \le b \\ 0 & x > b \end{array}\right.$$

$g'(a^-) = 0 \qquad g'(b^-) = f(b)$

$g'(a^+) = f(a) \qquad g'(b^+) = 0$

चूंकि f(x) का सहप्रान्त [1, ∞) f(a) तथा f(b) शून्य नही हो सकते है

अतः x = a व x = b पर यह अवकलनीय नही है।

21.* Let f: (0, ∞) → R be given by $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$. Then

[JEE (Advanced) 2014, Paper-1, (3, 0)/60]

(A) f(x) is monotonically increasing on [1, ∞)

(B) f(x) is monotonically decreasing on (0, 1)

(C) $f(x) + f\left(\frac{1}{x}\right) = 0$, for all x $\in$ (0, ∞)

(D) $f(2^x)$ is an odd function of x on R

माना कि f: (0, ∞) → R निम्न के द्वारा $f(x) = \int_{\frac{1}{x}}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$ परिभाषित है। तब

(A) [1, ∞) पर f(x) एकदिष्ट वर्धमान (monotonically increasing) है।

(B) (0, 1) पर f(x) एकदिष्ट ह्रसमान (monotonically increasing) है।

(C) सभी x $\in$ (0, ∞) के लिये, $f(x) + f\left(\frac{1}{x}\right) = 0$

(D) R पर $f(2^x)$, x का एक विषम फलन(odd function) है।

Ans. **(ACD)**

Sol. $f(x) = \int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$

$f'(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{x}{x^2} e^{-\left(x+\frac{1}{x}\right)}$

$f'(x) = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$

(A) For $x \in [1, \infty)$ $f'(x) > 0$ so (A) is correct.

(B) Obvious wrong.

(C) $f(x) + f(1/x) = \int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} + \underbrace{\int\limits_{x}^{1/x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt}_{\text{put } t=\frac{1}{p}}$

$\int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} - \int\limits_{1/x}^{x} e^{-\left(p+\frac{1}{p}\right)} \frac{dp}{p}$

$= 0$

(C) is correct

(D) Since $f(x) = -f\left(\frac{1}{x}\right)$

$f(2^x) = -f\left(\frac{1}{2^x}\right)$

$f(2^x) = -f(2^{-x})$

odd.

(D) is correct

ACD is answer

Hindi $f(x) = \int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$

$f'(x) = \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{x}{x^2} e^{-\left(x+\frac{1}{x}\right)}$

$f'(x) = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$

(A) For $x \in [1, \infty)$ $f' > 0$ अतः (A) सही है।

स्पष्टतः (B) गलत है।.

(C) $f(x) + f(1/x) = \int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} + \underbrace{\int\limits_{x}^{1/x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt}_{\text{put } t=\frac{1}{p}}$

$\int\limits_{1/x}^{x} e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t} - \int\limits_{1/x}^{x} e^{-\left(p+\frac{1}{p}\right)} \frac{dp}{p}$

$= 0$

(C) सही है

(D) चूंकि $f(x) = -f\left(\frac{1}{x}\right)$

$f(2^x) = -f\left(\frac{1}{2^x}\right)$

$f(2^x) = -f(2^{-x})$

(D) सही है।

ACD सही उत्तर है।

22. The value of $\int\limits_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$ is

[XII]

निम्न $\int\limits_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}$ dx का मान है :

[Definite Integration & its application] **[JEE (Advanced) 2014, Paper-1, (3, 0)/60]**

Ans. **(2)**

Sol. $4x^3 . \frac{d}{dx}\left(1-x^2\right)^5\Big|_0^1 - 12\int\limits_0^1 x^2.\frac{d}{dx}\left(1-x^2\right)^5 dx$

$= -12\left[\left(x^2.\left(1-x^2\right)^5\right)_0^1 - 2\int\limits_0^1 x.\left(1-x^2\right)^5 dx\right]$

$= 12\int\limits_0^1 2x\left(1-x^2\right)^5 dx$

$= -12\int\limits_1^0 t^5 dt = \frac{12}{6}\left(t^6\right)_0^1 = 2.$

Alternative : वैकल्पिक हल :

$\int_0^1 4x^3\left\{\frac{d^2}{dx^2}(1-x^2)^5\right\}dx$

$\frac{d}{dx}\left(\frac{d(1*-x^2)^5}{dx}\right) = \frac{d}{dx}\ (5(1-x^2)^4(-3x))$

$= -10\ \frac{d}{dx}\ (x(1-x^2)^4)$

$= -10[(1-x^2)^4 + x^4(1-x^2)^3(-2x)]$

$= [-10(1-x^2)^3 [1-x^2-8x^2]$

Hence Integral अतः समाकल

$= -40\ \int_0^1 x^3(1-x^2)^3(1-9x^2)dx$ Put x = sin θ रखने पर

$= -40\ \int_0^{\pi/2} \sin^3\theta\cos^7\theta d\theta + 360\ \int_0^1 \sin^5\theta\cos^7\theta\ d\theta$

$= -40.\ 1.\ \frac{2.6.4.2}{10.8.6.4.2} + 360\ .\ \frac{4.2.6.4.2}{12.10.8.6.4.2}$

$= -1 + 3 = 2$ Ans.

23. The following integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\text{cosecx})^{17}\,dx$ is equal to **[XII]**

[Definite Integration & its application] **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

निम्न समाकल (integral) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\text{cosecx})^{17}\,dx$ नीचे दिये गये विकल्पों में से किसके समान है ?

[Definite Integration & its application] **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

(A) $\int_{0}^{\log(1+\sqrt{2})} 2(e^{u}+e^{-u})^{16}du$

(B) $\int_{0}^{\log(1+\sqrt{2})} (e^{u}+e^{-u})^{17}du$

(C) $\int_{0}^{\log(1+\sqrt{2})} (e^{u}-e^{-u})^{17}du$

(D) $\int_{0}^{\log(1+\sqrt{2})} 2(e^{u}-e^{-u})^{16}du$

Ans. **(A)**

Sol. $I = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(2\text{cosecx})^{17}dx$

Put $\ell n \tan x/2 = t$ रखने पर $\Rightarrow \tan\frac{x}{2} = e^{t}$

$\Rightarrow \sin x = \frac{2e^{t}}{1+e^{2t}}$

$\text{cosec}\,x = \frac{e^{t}+e^{-t}}{2}$

$$I = 2\int_{\ell n(\sqrt{2}-1)}^{0}(e^{t}+e^{-t})^{16}.dt$$

$$= 2\int_{-\ell n(\sqrt{2}+1)}^{0}(e^{t}+e^{-t})^{16}.dt$$

since$(e^{t}+e^{-t})^{16}$ is an even function

चूंकि $(e^{t}+e^{-t})^{16}$ एक समफलन है।

$$\int_{-a}^{0} = \int_{0}^{a}$$

Hence अतः $I = \int_{0}^{\ell n(\sqrt{2}+1)} 2(e^{t}+e^{-t})^{16}dt$

Paragraph For Questions 24 and 25 (प्रश्न संख्या 24 और 25 के लिए अनुच्छेद)

Given that for each a ∈ (0, 1)

$$\lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}\ dt$$

exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0, 1).

दिया गया है कि प्रत्येक a ∈ (0, 1) के लिए सीमा

$$\lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}\ dt$$

वास्तव में है। माना कि यह सीमा g(a) है इसके अतिरिक्त यह भी दिया गया है कि अंतराल (interval) (0, 1) पर फलन g(a) अवकलनीय है।

[Definite Integration] **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

24. The value of $g\left(\frac{1}{2}\right)$ is **[XII]**

$g\left(\frac{1}{2}\right)$ का मान है–

(A*) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

Ans. **(A)**

Sol. $g'(a) = \int_h^{1-h} \frac{\partial}{\partial a} t^{-a}(1-t)^{a-1}dt$

$$= -\int_h^{1-h} t^{-a}(1-t)^{a-1}\ dt + t^{-a}\left(1-t\right)^{a-1}\ \ dt\ = 0$$

g(a) = constantअचर ⇒ g(a) = λ

$$g(a) = \underset{h\to 0^+}{Lt}\int_h^{1-h}\frac{1}{\sqrt{t(1-t)}}dt$$

$$= \int_h^{1-h}\frac{dt}{\sqrt{-\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}} = \left(\sin^{-1}\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)_h^{1-h}$$

$= \sin^{-1}(2t-1)\ \Big|_h^{1-h}$

$= \sin^{-1}(1-2h) - \sin^{-1}(2h-1)$

$= \pi$

25. The value of $g'\left(\frac{1}{2}\right)$ is **[XII]**

$g'\left(\frac{1}{2}\right)$ का मान है–

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0

Ans. **(D)**

Sol. $g(a) = \lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}dt$

$$g(1-a) = \lim_{h\to 0^+} \int_h^{1-h} t^{-(1-a)}(1-t)^{(1-a)-1}dt$$

$$= \lim_{h\to 0^+} \int_h^{1-h} t^{a-1}(1-t)^{-a}dt$$

$$= \lim_{h\to 0^+} \int_h^{1-h} (1-t)^{a-1}(1-(1-t))^{-a}\, dt \qquad \left\{\text{by } \int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx\right\}$$

$$= \lim_{h\to 0^+} \int_h^{1-h} (1-t)^{a-1}\ t^{-a}\ dt$$

$g(1 - a) = g(a)$

26. **List I** **List II**

[XII]

[Definite Integration & its application] **[JEE (Advanced) 2014, Paper-2, (3, –1)/60]**

P. The number of polynomials f(x) with non-negative integer coefficients of degree ≤ 2, satisfying f(0) = 0 and $\int_0^1 f(x)dx = 1$, is **1.** 8

Q. The number of points in the interval $\left[-\sqrt{13}, \sqrt{13}\right]$ at which f(x) = sin(x²) + cos(x²) attains its maximum value, is **2.** 2

R. $\int_{-2}^{2} \frac{3x^2}{(1+e^x)}$ dx equals **3.** 4

S. $\dfrac{\left(\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\int_{0}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}$ equals **4.** 0

सूची- I **सूची - II**

[XII]

P. अऋणात्मक पूर्णांक गुणांक (non-negative integer) वाले बहुपदों (polynomials), f(x), जिनकी घात (degree) ≤ 2 है, तथा जो f(0) = 0 एवम् $\int_0^1 f(x)dx = 1$ को संतुष्ट करती है, की संख्या है– **1.** 8

Q. अन्तराल $\left[-\sqrt{13}, \sqrt{13}\right]$ में स्थित उन बिन्दुओं की संख्या जिन पर f(x) = sin(x²) + cos(x²) का मान अधिकतम है, है– **2.** 2

R. $\int_{-2}^{2} \frac{3x^2}{(1+e^x)}$ dx का मान है– **3.** 4

S. $\dfrac{\left(\int_{-1/2}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}{\left(\int_{0}^{1/2} \cos 2x \log\left(\frac{1+x}{1-x}\right)dx\right)}$ का मान है–

	P	**Q**	**R**	**S**
(A)	3	2	4	1
(B)	2	3	4	1
(C)	3	2	1	4
(D)	2	3	1	4

Ans. **(D)**

Sol. (P) Let f(x) = $ax^2 + bx$, a, b $\in$ W (as f(0) = 0)

$$\int_0^1 ax^2 + bc = \frac{a}{3} + \frac{b}{2} = 1 \Rightarrow \quad 2a + 3b = 6$$

$\Rightarrow$ (a, b) $\equiv$ (3, 0), (0, 2)

Number of such polynomials = 2

(Q) $f(x) = \sqrt{2}\sin\left(x^2 + \frac{\pi}{4}\right)$

$x^2 + \frac{\pi}{4} = 2n\pi + \frac{\pi}{2}$ if f(x) is maximum

$x^2 = 2n\pi + \frac{\pi}{2}$

for n = 0, 1 $\quad x^2 \in [0, 13]$

(R) $$\int_{-2}^{2} \frac{3x^2}{1+e^x}dx = \int_0^2 3x^2\left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right)dx \left\{\int_{-a}^{a} f(x)dx = \int_0^a (f(x)+f(-x))dx\right\}$$

$$= \int_0^2 3x^2\left(\frac{1}{1+e^x} + \frac{e^x}{1+e^x}\right)dx = \int_0^2 3x^2 dx = x^3\Big|_0^2 = 8$$

(S) $$\int_{-1/2}^{1/2} \cos 2x \, \ell n\left(\frac{1+x}{1-x}\right)dx = 0$$ (as it is an odd function)

Hence P $\to$ 2, Q $\to$ 3, R $\to$ 1, S $\to$ 4

(D) Ans.

Hindi (P) माना f(x) = $ax^2 + bx$, a, b $\in$ W (चूंकि f(0) = 0)

$$\int_0^1 ax^2 + bc = \frac{a}{3} + \frac{b}{2} = 1 \Rightarrow \quad 2a + 3b = 6$$

$\Rightarrow$ (a, b) $\equiv$ (3, 0), (0, 2)

इस प्रकार बहुपदों की संख्या = 2

(Q) $f(x) = \sqrt{2}\sin\left(x^2 + \frac{\pi}{4}\right)$

$x^2 + \frac{\pi}{4} = 2n\pi + \frac{\pi}{2}$ यदि f(x) अधिकतम है।

$x^2 = 2n\pi + \frac{\pi}{2}$

n = 0, 1 के लिए $x^2 \in [0, 13]$

(R) $$\int_{-2}^{2} \frac{3x^2}{1+e^x}dx = \int_0^2 3x^2\left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right)dx \left\{\int_{-a}^{a} f(x)dx = \int_0^a (f(x)+f(-x))dx\right\}$$

$$= \int_0^2 3x^2\left(\frac{1}{1+e^x}+\frac{e^x}{1+e^x}\right)dx = \int_0^2 3x^2dx = x^3\Big|_0^2 \;= 8$$

(S) $\int_{-1/2}^{1/2} \cos 2x\,\ell n\left(\frac{1+x}{1-x}\right)dx \;= 0$ (चूंकि यह एक विषम फलन है)

अतः $P \to 2, Q \to 3, R \to 1, S \to 4$

(D) Ans.

27. Let f: R → R be a function defined by f(x) = $\begin{cases}[x], & x \le 2 \\ 0, & x > 2\end{cases}$ where [x] is the greatest integer less than or equal to x. If I = $\int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}$ dx , then the value of (4I–1) is

f(x) = $\begin{cases}[x], & x \le 2 \\ 0, & x > 2\end{cases}$ से परिभाषित है, जहाँ [x], x से कम या x के बराबर के महत्तम पूर्णांक (greatest integer less than or equal to x) को दर्शाता है। यदि I = $\int_{-1}^{2} \frac{xf(x^2)}{2+f(x+1)}$ dx , तब (4I–1) का मान है।

[Definite Integration] **[JEE (Advanced) 2015, P-1 (4, 0) /88]**

Ans. **0**

Sol. $I = \int_{-1}^{2} \frac{x[x^2]}{2+[x+1]}dx = \int_{-1}^{2} \frac{x[x^2]}{3+[x+1]}dx = \int_{-1}^{0} \frac{0}{3-1}dx + \int_{0}^{1} \frac{0}{3+0}dx + \int_{1}^{\sqrt{2}} \frac{x.1}{3+1}dx$

$$= \frac{1}{4}\left[\frac{x^2}{2}\right]_1^{\sqrt{2}} = \frac{2-1}{8} = \frac{1}{4} \qquad \therefore \qquad 4I - 1 = 0$$

28. If α = $\int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$ where tan⁻¹ x takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is **[JEE (Advanced) 2015, P-2 (4, 0) / 80]**

यदि α = $\int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$ जहाँ tan⁻¹ x केवल मुख्य मानों (principal values) को लेता है, तब $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ का मान है

Ans. **9**

Sol. $\alpha = \int_0^1 e^{9x+3\tan^{-1}x}.\left(\frac{12+9x^2}{1+x^2}\right)$ dx

⇒ $\alpha = \left(e^{9x+3\tan^{-1}x}\right)_0^1$

⇒ $\alpha = e^{9+\frac{3\pi}{4}} - 1$

⇒ $\ell n\,(1 + \alpha) = 9 + \frac{3\pi}{4}$

Aliter :

$\alpha = \int_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Let माना $9x + 3\tan^{-1}x = t$

$$\Rightarrow \quad \left(9+\frac{3}{1+x^2}\right)dx = dt \quad \Rightarrow \quad \left(\frac{12+9x^2}{1+x^2}\right)dx = dt$$

$$\Rightarrow \quad \alpha = \int_0^{9+3\pi/4} e^t dt = \left(e^t\right)_0^{9+3\pi/4} = e^{9+3\pi/4} - 1$$

Now अब $\log_e|1 + \alpha| - 3\pi/4 = \log_e e^{(9+3\pi/4)} - 3\pi/4 = 9$

29. Let f: R → R be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that $F(x) = \int_{-1}^{x} f(t)\ dt$ for all $x \in [-1, 2]$ and $G(x) = \int_{-1}^{x} t\,|f(f(t))|\ dt$ for all $x \in [-1, 2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is.

[JEE (Advanced) 2015, P-2 (4, 0) / 80] **(Moderate)**

माना कि f: R → R एक संतत विषम फलन है जिसका मान केवल एक बिन्दु पर ही शून्य होता है तथा $f(1) = \frac{1}{2}$ है। माना कि सभी $x \in [-1, 2]$ के लिए $F(x) = \int_{-1}^{x} f(t)\ dt$ एवं सभी $x \in [-1, 2]$ के लिए $G(x) = \int_{-1}^{x} t\,|f(f(t))|\ dt$ है। यदि $\lim_{x\to 1}\frac{F(x)}{G(x)} = \frac{1}{14}$ है, तब $f\left(\frac{1}{2}\right)$ का मान है

Ans. 7 **[Definite Integration][JEE (Advanced) 2015, P-2 (4, 0) / 80]**

Sol. $F(x) = \int_{-1}^{x} f(t)dt = \int_{1}^{x} f(t)dt$

$G(x) = t|f(f(t))|dt = t|f(f(t))|dt$

$\lim_{x\to 1}\frac{F(x)}{G(x)}$

L' hospitals (L' हॉस्पिटल) $\lim_{x\to 1}\frac{f(x)}{x\,|\,f\left(f\left(x\right)\right)|} = \frac{1}{14}$

$$\frac{\frac{1}{2}}{1\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$$

$f\left(\frac{1}{2}\right) = 7$

30*. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is (are)

माना कि सभी $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ के लिए, $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ है, तब सही कथन है (हैं)

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A*) $\int_0^{\pi/4} x f(x)\ dx = \frac{1}{12}$ (B*) $\int_0^{\pi/4} f(x)\ dx = 0$ **(Tough)**

(C) $\int_0^{\pi/4} xf(x)\ dx = \frac{1}{6}$ (D) $\int_0^{\pi/4} f(x)\ dx = 1$ **[Definite Integration]**

Ans. **(A,B)**

Sol. $f(x) = (7\tan^6 x - 3\tan^2 x).\sec^2 x$

$\therefore \quad \int_0^{\frac{\pi}{4}} f(x)\,dx = \int_0^1 \left(7t^6 - 3t^2\right)\,dt = (t^7 - t^3)_0^1 = 0$ **Ans. (B)**

Now अब $\int_0^{\frac{\pi}{4}} xf(x)dx = \int_0^1 \frac{\left(7t^6 - 3t^2\right)}{\text{II}}\frac{\tan^{-1}t}{\text{I}}\ dt$

$$= \left(\tan^{-1}t.\left(t^7 - t^3\right)\right)_0^1 - \int_0^1 \left(t^7 - t^3\right).\frac{1}{1+t^2}dt$$

$$= \int_0^1 \frac{t^3\left(1-t^4\right)}{1+t^2}\ dt = \int_0^1 t^3\left(1-t^2\right)\ dt$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

31. Let $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \le \int_{1/2}^{1} f(x)\ dx \le M$, then the possible values of m and M are

(Tough)

माना कि सभी $x \in R$ के लिए, $f'(x) = \frac{192x^3}{2+\sin^4 \pi x}$ एवं $f\left(\frac{1}{2}\right) = 0$ है। यदि $m \le \int_{1/2}^{1} f(x)\ dx \le M$, तब m और M के सही संभव मान है (हैं)

[Definite Integration]

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A) m = 13, M = 24   (B) $m = \frac{1}{4}, M = \frac{1}{2}$

(C) m = – 11, M = 0   (D*) m = 1, M = 12

Ans. **(D)**

Sol. $f'(x) = \frac{192x^3}{2+\sin^4(\pi x)} \quad \forall x \in R\ ;\ f\left(\frac{1}{2}\right) = 0$

Now अब $64x^3 \le f'(x) \le 96x^3\ \forall x \in \left[\frac{1}{2},1\right]$

So इसलिए $16x^4 - 1 \le f(x) \le 24x^4 - \frac{3}{2}\ \ \forall x \in \left[\frac{1}{2},1\right]$

$$\frac{16}{5}.\frac{31}{32} - \frac{1}{2} \le \int_{1/2}^{1} f(x)dx \le \frac{24}{5}.\frac{31}{32} - \frac{3}{4}$$

$\Rightarrow \quad \frac{26}{10} \le \int_{1/2}^{1} f(x)dx \le \frac{78}{20}$ hence अत: (D)

(A) is incorrect as सही नहीं है जैसा कि $\int_{1/2}^{1} f(x)dx \le \frac{78}{20}$

(B) is incorrect as सही नहीं है जैसा कि $\int_{1/2}^{1} f(x)dx > \frac{26}{10}$

(C) is incorrect as सही नहीं है जैसा कि $\int_{1/2}^{1} f(x)dx > 0$

32*. The option(s) with the values of a and L that satisfy the following equation is(are)

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L \ ?$$

(Tough) **[Definite Integration]**

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

(A*) a = 2, L = $\frac{e^{4\pi}-1}{e^{\pi}-1}$

(B) a = 2, L = $\frac{e^{4\pi}+1}{e^{\pi}+1}$

(C*) a = 4, L = $\frac{e^{4\pi}-1}{e^{\pi}-1}$

(D) a = 4, L = $\frac{e^{4\pi}+1}{e^{\pi}+1}$

निम्नलिखित में से a और L के कौन सा (से) मान समीकरण

$$\frac{\int_0^{4\pi} e^t(\sin^6 at + \cos^4 at)dt}{\int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt} = L$$

को संतुष्ट करता (करते) हैं ? **[JEE (Advanced) 2015, P-2 (4, –2)/ 80]**

(A*) a = 2, L = $\frac{e^{4\pi}-1}{e^{\pi}-1}$

(B) a = 2, L = $\frac{e^{4\pi}+1}{e^{\pi}+1}$

(C*) a = 4, L = $\frac{e^{4\pi}-1}{e^{\pi}-1}$

(D) a = 4, L = $\frac{e^{4\pi}+1}{e^{\pi}+1}$

Ans. **(A,C)**

Sol. $I_1 = \int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt + \int_{\pi}^{2\pi} e^t(\sin^6 at + \cos^4 at)dt + \int_{2\pi}^{3\pi} e^t(\sin^6 at + \cos^4 at)dt + \int_{3\pi}^{4\pi} e^t(\sin^6 at + \cos^4 at)dt$

$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi}) \int_0^{\pi} e^t(\sin^6 at + \cos^4 at)dt$

$\Rightarrow \frac{I_1}{I_2} = 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi}-1}{e^{\pi}-1}$

Paragraph For Questions 33 and 34
(प्रश्न संख्या 33 और 34 के लिए अनुच्छेद)

Let F : R → R be a thrice differentiable function. Supose that F(1) = 0, F(3) = –4 and F'(x) < 0 for all x ∈ (1/2, 3). Let f(x) = xF(x) for all x ∈ R. **[Definite Integration]**

माना कि F : R → R एक फलन है जो तीन बार अवकलनीय (thrice differentiable) है। माना कि F(1) = 0, F(3) = –4 और सभी x ∈ (1/2, 3) के लिए F'(x) < 0 है। माना कि सभी x ∈ R के लिए, f(x) = xF(x) है।

[Definite Integration]

[JEE (Advanced) 2015, P-2 (4, –2)/ 80]

33*. The correct statement(s) is(are)

(A*) f '(1) < 0

(B*) f (2) < 0

(C*) f '(x) ≠ 0 for any x ∈ (1, 3)

(D) f '(x) = 0 for some x ∈ (1, 3)

निम्नलिखित में से सही कथन है (हैं) **(Tough)**

(A*) f '(1) < 0

(B*) f (2) < 0

(C*) किसी भी $x \in (1, 3)$ के लिए $f '(x) \neq 0$ (D) कुछ $x \in (1, 3)$ के लिए $f '(x) = 0$

Ans. **(A,B,C)**

Sol. $f'(x) = xf'(x) + f(x)$

$\Rightarrow \quad f'(1) = f'(1) + f(1) = f'(1) < 0 \quad \Rightarrow \quad (A)$

$f(2) = 2f(2) < 0 \quad \Rightarrow \quad (B)$

for $x \in (1, 3)$ के लिए, $f'(x) = xf'(x) + f(x) < 0 \quad \Rightarrow \quad (C)$

34*. If $\int_1^3 x^2 \ F'(x)dx = -12$ and $\int_1^3 x^3 \ F''(x)dx = 40$, then the correct expression(s) is(are)

यदि $\int_1^3 x^2 \ F'(x)dx = -1$: और $\int_1^3 x^3 \ F''(x)dx = 40$ है, तब सही कथन है (हैं) **(Tough)**

(A) $9f '(3) + f '(1) - 32 = 0$ (B) $\int_1^3 f(x)dx = 12$

(C*) $9f '(3) - f '(1) + 32 = 0$ (D*) $\int_1^3 f(x)dx = -12$

Ans. **(C,D)**

Sol. $\int_1^3 x^3 f''(x)dx = 40 \quad \Rightarrow \quad \left[x^3 f'(x)\right]_1^3 - \int_1^3 3x^2 f'(x)dx = 40$

$\Rightarrow \quad \left[x^2 f'(x) - xf(x)\right]_1^3 - 3(-12) = 40$

$\Rightarrow \quad 9f'(3) - 3f(3) - f'(1) + f(1) = 4$

$\Rightarrow \quad 9f'(3) + 36 - f'(1) + 0 = 4 \quad \Rightarrow \quad 9f'(3) - f'(1) + 32 = 0 \quad \Rightarrow \quad (C)$

$\Rightarrow \quad \int_1^3 x^2 f'(x)dx = -12 \quad \Rightarrow \quad \left[x^2 f(x)\right]_1^3 - \int_1^3 2xf(x)dx = -12$

$\Rightarrow \quad -36 - 2\int_1^3 f(x)dx = -12 \quad \Rightarrow \quad \int_1^3 f(x)dx = -12 \quad \Rightarrow \quad (D)$

35. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t \ \ dt$ for all $x \in R$ and $f: \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in, \left[0, \frac{1}{2}\right]$ if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is

माना कि सभी $x \in R$ के लिए, $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t \ \ dt$ तथा $f: \left[0, \frac{1}{2}\right] \to [0, \infty)$ एक संतत फलन है। यदि उन सभी $a \in \left[0, \frac{1}{2}\right]$ के लिए $F'(a) + 2$ उस क्षेत्र का क्षेत्रफल है, जो कि $x = 0$, $y = 0$, $y = f(x)$ और $x = a$, से घिरा (bounded) हुआ है, तब $f(0)$ का मान है। **[JEE (Advanced) 2015, P-1 (4, 0) /88]**

Ans. **3**

Sol. $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 + dt$; $\quad F'(x) = 2\left(\cos\left(x^2 + \frac{\pi}{6}\right)\right)^2 2x - 2\cos^2 x$

$\because \quad F'(a) + 2 = \int_0^a f(x)dx \quad \Rightarrow \quad 2\left(\cos\left(a^2 + \frac{\pi}{6}\right)\right)^2 2a - 2\cos^2 a + 2 = \int_0^a f(x)dx$

$\Rightarrow \quad 4\cos^2\left(a^2 + \frac{\pi}{6}\right) + 4a\ 2\cos\left(a^2 + \frac{\pi}{6}\right).\left(-\sin\left(a^2 + \frac{\pi}{6}\right)\right) \times 2a + 4\cos a \sin a = f(a)$

$\therefore \qquad f(0) = 4\left(\frac{\sqrt{3}}{2}\right)^2 = 3$

36. The total number of distinct x ∈ (0, 1] for which $\int_0^x \frac{t^2}{1+t^4}dt = 2x-1$ is

[JEE (Advanced) 2016, Paper-1, (3, 0)/62]

ऐसे सभी भिन्न (distinct) x ∈ (0, 1], जिनके लिए $\int_0^x \frac{t^2}{1+t^4}dt = 2x-1$ है, की कुल संख्या है–

[JEE (Advanced) 2016, Paper-1, (3, 0)/62]

Ans. 1

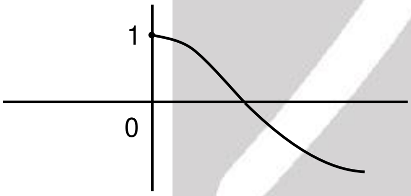
Sol. Let $f(x) = \int_0^x \frac{t^2}{1+t^4}dt + 1 - 2x$

$f'(x) = \frac{x^2}{1+x^4} - 2$ is negative

< 0

$f(0) = 1$ and

$f(1) = \underbrace{\int_0^1 \frac{t^2}{1+t^4}dt}_{\text{Less than one}} - 1$



37. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+e^x}dx$ is equal to **[JEE (Advanced) 2016, Paper-2, (3, –1)/62]**

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+e^x}dx$ का मान है– **[JEE (Advanced) 2016, Paper-2, (3, –1)/62]**

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$ (C) $\pi^2 - e^{\pi/2}$ (D) $\pi^2 + e^{\pi/2}$

Ans. **(A)**

Sol. $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{(1+e^x)} dx \quad \Rightarrow \quad I = \int_{0}^{\pi/2} \left(\frac{x^2 \cos x}{1+e^x} + \frac{x^2 \cos x}{1+e^{-x}} \right) dx$

$I = \int_{0}^{\pi/2} x^2 \cos x dx = (x^2 \sin x - 2x(-\cos x) + (2)(-\sin x))_0^{\pi/2} \quad = \left(\frac{\pi^2}{4} - 2 \right) - (0) = \frac{\pi^2}{4} - 2$

38. Area of the region $\left\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq x + 9 \leq 15\right\}$ is equal to

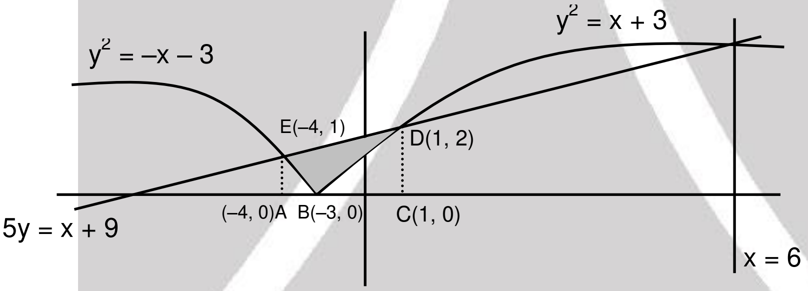
[JEE (Advanced) 2016, Paper-2, (3, 1)/62]

क्षेत्र (region) $\left\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq x + 9 \leq 15\right\}$ का क्षेत्रफल (area) है–

[JEE (Advanced) 2016, Paper-2, (3, 1)/62]

(A) $\frac{1}{6}$ (B) $\frac{4}{3}$ (C) $\frac{3}{2}$ (D) $\frac{5}{3}$

Ans. **(C)**



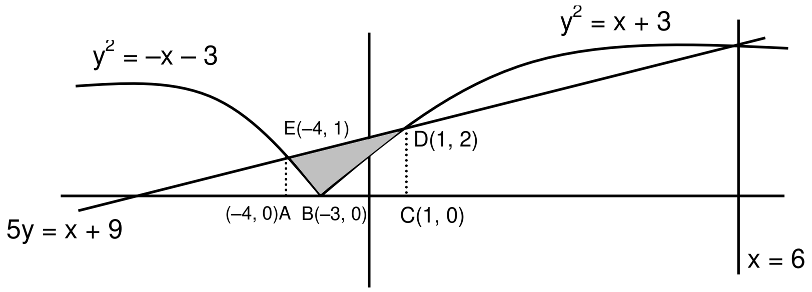
Area ABE (under parabola) = $\int_{-4}^{-3} \sqrt{-x-3}\, dx = \frac{2}{3}$

Area BCD (under parabola) = $\int_{-3}^{1} \sqrt{x+3}\, dx = \frac{16}{3}$

Area of trapezium ACDE = $\frac{1}{2}(1 + 2)5 = \frac{15}{2}$

Required area = $\frac{15}{2} - \frac{16}{3} - \frac{2}{3} = \frac{3}{2}$

Hindi



ABE का क्षेत्रफल (परवलय के अन्दर) = $\int_{-4}^{-3} \sqrt{-x-3}\, dx = \frac{2}{3}$

BCD का क्षेत्रफल (परवलय के अन्दर) = $\int_{-3}^{1} \sqrt{x+3}\, dx = \frac{16}{3}$

समलम्ब चतुर्भुज ACDE का क्षेत्रफल = $\frac{1}{2}(1 + 2)5 = \frac{15}{2}$

अभीष्ट क्षेत्रफल = $\frac{15}{2} - \frac{16}{3} - \frac{2}{3} = \frac{3}{2}$

39*. Let f(x) = $\lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$, for all x > 0. Then

[JEE (Advanced) 2016, Paper-2, (4, –2)/62]

माना कि सभी $x > 0$ के लिए f(x) = $\lim_{n\to\infty}\left(\frac{n^n(x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{\frac{x}{n}}$ है। तब

[JEE (Advanced) 2016, Paper-2, (4, –2)/62]

(A) $f\left(\frac{1}{2}\right) \ge f(1)$ (B*) $f\left(\frac{1}{3}\right) \le f\left(\frac{2}{3}\right)$ (C*) $f'(2) \le 0$ (D) $\frac{f'(3)}{f(3)} \ge \frac{f'(2)}{f(2)}$

Ans. **(B,C)**

Sol. $$f(x) = \lim_{n\to\infty}\left\{\frac{n^{2n}\left(\frac{x}{n}+1\right)\left(\frac{x}{n}+\frac{1}{2}\right)\ldots\ldots\left(\frac{x}{n}+\frac{1}{n}\right)}{n!\,n^{2n}\left(\frac{x^2}{n^2}+1\right)\left(\frac{x^2}{n^2}+\frac{1}{2^2}\right)\ldots\ldots\left(\frac{x^2}{n^2}+\frac{1}{n^2}\right)}\right\}^{x/n}$$

$$\ell nf(x) = \lim_{n\to\infty}\frac{x}{n}\left\{\sum_{r=1}^{n}\ell n\left(\frac{x}{n}+\frac{1}{r}\right)-\sum_{r=1}^{n}\ell n\left(\frac{rx^2}{n^2}+\frac{1}{r}\right)\right\} = \lim_{n\to\infty}\frac{x}{n}\left\{\sum_{r=1}^{n}\ell n\left(1+\frac{rx}{n}\right)-\sum_{r=1}^{n}\ell n\left(1+\frac{r^2x^2}{n^2}\right)\right\}$$

$$\ell n(f(x)) = x\int_0^1 \ell n\,(1 + xy)dy - x\int_0^1 \ell n\,(1 + x^2y^2)dy$$

Let xy = t

$$\ell n(f(x)) = \int_0^x \ell n\,(1 + t)dt - \int_0^x \ell n\,(1 + t^2)dt$$

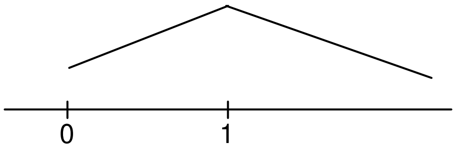
$$\frac{f'(x)}{f(x)} = \ell n\left(\frac{1+x}{1+x^2}\right)$$

$$\frac{f'(2)}{f(2)} = \ell n\left(\frac{3}{5}\right) < 0 \qquad \Rightarrow \qquad f'(2) < 0$$

$$\frac{f'(3)}{f(3)} = \ell n\left(\frac{4}{10}\right) = \ell n\left(\frac{2}{5}\right) \qquad \Rightarrow \qquad \frac{f'(2)}{f(2)} \geq \frac{f'(3)}{f(3)}$$

Now $\frac{f'(x)}{f(x)} > 0$ in (0, 1) and $\frac{f'(x)}{f(x)} < 0$ in $(1, \infty)$

अब (0, 1) में $\frac{f'(x)}{f(x)} > 0$ तथा $(1, \infty)$ में $\frac{f'(x)}{f(x)} < 0$



f(x) is increasing in (0, 1) & decreasing in $[1, \infty)$ (as f(x) is positive)

f(x), (0, 1) में वर्द्धमान तथा $[1, \infty)$ में ह्रासमान (क्योंकि f(x) धनात्मक है।)

hence अतः $f(1) \geq f\left(\frac{1}{2}\right)$ and तथा $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$

40*. Let $f : R \to (0, 1)$ be a continuous function. Then, which of the following function(s) has (have) the value zero at some point in the interval (0, 1) ? **[JEE(Advanced) 2017, Paper-1,(4, –2)/61]**

माना कि $f : R \to (0, 1)$ एक सतत् फलन (continuous function) है। तब निम्न फलनों में से कौन से फलन (नों) का (के) मान अन्तराल (interval) (0, 1) के किसी बिन्दु पर शून्य होगा **Definite integration**

(A) $e^x - \int_0^x f(t) \sin t \, dt$

(B) $f(x) + \int_0^{\frac{\pi}{2}} f(t) \sin t \, dt$

(C*) $x - \int_0^{\frac{\pi}{2}-x} f(t) \cos t \, dt$

(D*) $x^9 - f(x)$

Ans. **(C,D)**

Sol. $e^x \in (1,e)$ for $x \in (0,1)$ के लिए

and और $0 < \int_0^x f(t)\sin t\,dt < 1$ in (0,1) $\Rightarrow$ (A) is wrong (असत्य है)

$f(x) + \int_0^{\pi/2} f(t)\sin t\,dt > 0$ $\Rightarrow$ (B) is wrong (असत्य है)

Let माना $g(x) = x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$ $\Rightarrow$ $g(0) = -\int_0^{\frac{\pi}{2}} f(t)\cos t\,dt < 0$

$g(1) = 1 - \int_0^{\frac{\pi}{2}-1} f(t)\cos t\,dt > 0$ $\Rightarrow$ (C) is correct (सत्य है)

Let माना $h(x) = x^9 - f(x)$

$h(0) = -f(0) < 0$

$h(1) = 1 - f(1) > 0$ $\Rightarrow$ (D) is correct (सत्य है)

41.✍ Let $f : R \to R$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$. If

$g(x) = \int_x^{\frac{\pi}{2}} [f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t \, f(t)]\,dt$ for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x\to 0} g(x) =$

[JEE(Advanced) 2017, Paper-1,(3, 0)/61] **Definite integration**

माना कि f : R → R इस प्रकार का अवकलनीय फलन (differentiable function) है कि $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ एवम् $f'(0) = 1$ है। यदि $x \in \left(0, \frac{\pi}{2}\right]$ के लिये $g(x) = \int_x^{\frac{\pi}{2}} [f'(t)\text{cosec}\,t - \cot t\ \text{cosec}\,t\ f(t)]\,dt$ है, तब $\lim_{x \to 0} g(x) =$

Ans. **(2)**

Sol. $g(x) = \int_x^{\pi/2} \frac{d}{dt}(f(t)\text{cosec}\,t)\,dt$

$g(x) = f\left(\frac{\pi}{2}\right)\text{cosec}\left(\frac{\pi}{2}\right) - f(x)\ \text{cosec}x$

$g(x) = 3 - f(x)\ \text{cosec}x$

$g(x) = 3 - \frac{f(x)}{\sin x}$

$\lim_{x \to 0} g(x) = 3 - \lim_{x \to 0} \frac{f(x)}{\sin x}$

$= 3 - \lim_{x \to 0} \frac{f'(x)}{\cos x} = 3 - \frac{1}{1} = 2$

42*. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then **[JEE(Advanced) 2017, Paper-2,(4, –2)/61]**

Definiate integration

यदि $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, तब

(A) $I > \log_e 99$ (B) $I < \log_e 99$ (C) $I < \frac{49}{50}$ (D) $I > \frac{49}{50}$

Ans. **(BD)**

Sol. Put $x - k = p$ रखने पर

$I = \sum_{k=1}^{98} \int_0^1 \frac{k+1}{(k+p)(k+p+1)} dp$

$I > \sum_{k=1}^{98} \int_0^1 \frac{k+1}{(k+p+1)^2} dp$

$I > \sum_{k=1}^{98} (k+1)\left(\frac{-1}{(k+p+1)}\right)_0^1$

$$I > \sum_{k=1}^{98}(k+1)\left(\frac{1}{k+1}-\frac{1}{k+2}\right)$$

$$I > \sum_{k=1}^{98}\frac{1}{k+2}=\frac{1}{3}+\text{.........}+\frac{1}{100}$$

$$I > \frac{1}{100}+\text{.....}+\frac{1}{100}=\frac{98}{100}$$

$$I > \frac{49}{50}$$

$$\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$$

$$\frac{k+1}{x(x+1)}<\frac{k+1}{x(k+1)}(\because \text{ least value of } x+1 \text{ is } k+1)$$

$$\Rightarrow \frac{k+1}{x(x+1)} < \frac{1}{x}$$

$$\Rightarrow I < \sum_{k=1}^{98}\int_k^{k+1}\frac{1}{x}dx$$

$$\Rightarrow I < \sum_{k=1}^{98}\ell n(k+1)-\ell nk \quad \Rightarrow I < \ell n99$$

43*. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in R^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then **[JEE(Advanced) 2017, Paper-2,(4, –2)/61]**

यदि रेखा $x = \alpha$ क्षेत्र (region) $R = \{(x, y) \in R^2 : x^3 \le y \le x, 0 \le x \le 1\}$ के क्षेत्रफल को दो बराबर भागों में विभाजित करती है, तब

Definiate integration

(A) $2\alpha^4 - 4\alpha^2 + 1 = 0$ (B) $\alpha^4 + 4\alpha^2 - 1 = 0$ (C) $\frac{1}{2} < \alpha < 1$ (D) $0 < \alpha \le \frac{1}{2}$

Ans. **(AC)**

Sol. $y = x^3$

$$\int_0^1 (x-x^3)dx = \frac{1}{2}-\frac{1}{4}=\frac{1}{4}$$

$$\int_0^\alpha (x-x^3)dx = \frac{1}{8}$$

$4\alpha^2 - 2\alpha^4 = 1$

$2\alpha^4 - 4\alpha^2 + 1 = 0$

$2t^2 - 4t + 1 = 0$ (taking $t = \alpha^2$ लेने पर)

$$t = \frac{4 \pm \sqrt{16-8}}{4}$$

$$t = \frac{4 \pm 2\sqrt{2}}{4}$$

$$t = \alpha^2 = 1 \pm \frac{1}{\sqrt{2}}$$

$$\therefore \qquad \alpha^2 = 1 - \frac{1}{\sqrt{2}} \qquad \Rightarrow \qquad \frac{1}{2} < \alpha < 1$$

44. If g(x) = $\int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\,dt$, then **[JEE(Advanced) 2017, Paper-2,(4, –2)/61]**

Definiate integration

यदि g(x) = $\int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\,dt$, तब

(A) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = -2\pi$ (C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(\frac{\pi}{2}\right) = -2\pi$

Ans. **(BONUS)**

Sol. $g(x) = \int_{\sin x}^{\sin 2x} \sin^{-1}(t)dt$

$g'(x) = \sin^{-1}(\sin 2x) . \cos 2x. 2 - \sin^{-1}(\sin x) . \cos x$

$= 2\cos 2x . \sin^{-1}(\sin 2x) - \cos x . \sin^{-1}(\sin x)$

$g'\left(-\frac{\pi}{2}\right) = 2\cos(-\pi) \sin^{-1}(\sin(-\pi)) - \cos\left(-\frac{\pi}{2}\right). \sin^{-1}(\sin\left(-\frac{\pi}{2}\right)) = 0$

$g'\left(\frac{\pi}{2}\right) = 2\cos(\pi) \sin^{-1}(\sin(\pi)) - \cos\left(\frac{\pi}{2}\right). \sin^{-1}(\sin\left(\frac{\pi}{2}\right)) = 0$

45. For each positive integer n, let $y_n = \frac{1}{n}((n+1)(n+2) \ldots (n+n))^{1/n}$.

For x ∈ R, let [x] be the greatest integer less than or equal to x. If $\lim_{n\to\infty} y_n = L$, then the value of [L] is

________. **[JEE(Advanced) 2018, Paper-1,(3, 0)/60]**

प्रत्येक धनात्मक पूर्णांक (positive integer) n के लिए, मानाकि $y_n = \frac{1}{n}((n+1)(n+2) \ldots (n+n))^{1/n}$.

x ∈ R के लिए मानाकि [x], x से छोटा या x के बराबर महत्तम पूर्णांक (greatest integer) है। यदि $\lim_{n\to\infty} y_n = L$, तब

[L] का मान है ________.

Ans. **(1)**

Sol. $y_n = \left(\frac{n+1}{n}\frac{n+2}{n}......\frac{n+n}{n}\right)^{\frac{1}{n}}$

$$\log L = \lim_{x\to\infty}\frac{1}{n}\sum_{r=1}^{n}\log\left(1+\frac{r}{n}\right)$$

$$= \int_0^1 \log(1+x)\,dx = \int_1^2 \log x\,dx = |x \log x - x|_1^2 = 2 \log 2 = \log \frac{4}{e}$$

$$\Rightarrow \quad L = \frac{4}{e} \quad \Rightarrow \quad [L] = 1$$

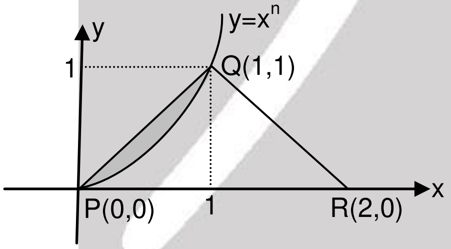
46.✍ A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n$ $(n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is **[Definite integration]**

[JEE(Advanced) 2018, Paper-1,(3, 0)/60]

एक किसान F_1 के पास एक त्रिभुजाकार (triangular) भूमि है जिसके शीर्ष (vertices) P(0, 0), Q(1, 1) और R(2, 0). पर है। एक पडौसी किसान F_2 इस भूमि से उस क्षेत्र को लेता है जो भुजा PQ और $y = x^n$ $(n > 1)$ के रूप वाले वक्र (curve) के बीच स्थित है। यदि किसान F_2 द्वारा लिए गये क्षेत्र (region) का क्षेत्रफल (area) ΔPQR के क्षेत्रफल का ठीक 30% है, तब n का मान है__________

Ans. **(4)**

Sol.



$$\int_0^1 (x-x^n)dx = \frac{3}{10}\left(\frac{1}{2}\times 2\times 1\right) \quad \Rightarrow \quad \left|\frac{x^2}{2}-\frac{x^{n+1}}{n+1}\right|_0^1 = \frac{3}{10} \quad \Rightarrow \quad \frac{1}{2}-\frac{1}{n+1} = \frac{3}{10}$$

$$\Rightarrow \quad \frac{1}{n+1} = \frac{1}{2}-\frac{3}{10} = \frac{1}{5} \quad \Rightarrow \quad n = 4$$

47. The value of the integral $\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{\frac{1}{4}}}dx$ is _____ . **[Definite integration]**

समाकल $\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{((x+1)^2(1-x)^6)^{\frac{1}{4}}}$ का मान है _____ । **[JEE(Advanced) 2018, Paper-2,(3, 0)/60]**

Ans. **(2)**

Sol. $$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$$

$$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\frac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$$

put $\frac{1-x}{1+x} = t$ रखने पर $\Rightarrow$ $\frac{-2dx}{(1+x)^2} = dt$

$$I = \int_1^{\frac{1}{3}} \frac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}} = \frac{-(1+\sqrt{3})}{2} \times \left|\frac{-2}{\sqrt{t}}\right|_1^{1/3} = (1+\sqrt{3})(\sqrt{3}-1) = 2$$

48. The area of the region $\{(x, y) : xy \le 8,\ 1 \le y \le x^2\}$ is **{Definite Inte [AR-TC]-M-302}**

क्षेत्र $\{(x, y) : xy \le 8,\ 1 \le y \le x^2\}$ का क्षेत्रफल (area) है -

(A) $16 \log_e 2 - 6$ (B) $8 \log_e 2 - \frac{7}{3}$ (C) $16 \log_e 2 - \frac{14}{3}$ (D) $8 \log_e 2 - \frac{14}{3}$

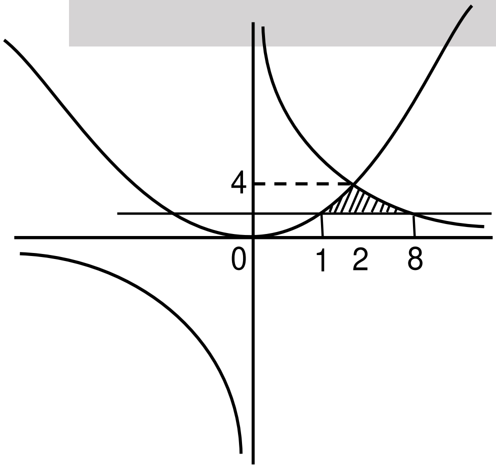
Ans. **(C)** **[JEE(Advanced) 2019, Paper-1,(4, –1)/62]**

Sol. $xy \le 8$

$1 \le y \le x^2$

$x^2 . x = 8$

$x = 2$



Required Area अभीष्ट क्षेत्रफल $= \int_1^4 \left(\frac{8}{y} - \sqrt{y}\right) dy = \left[8\ell n\, y - \frac{y^{3/2}}{3/2}\right]_1^4$

$= 8\,\ell n4 - \frac{2}{3}\,.\,8 - 0 + \frac{2}{3} = 16\ell n2 - \frac{14}{3}$

49. $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then find $27I^2$ equals ……. **[Definite integration]** **[E]**

यदि $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ तब $27I^2$ बराबर है **{[DI-SP]-E-302}**

Ans. **(4)** **[JEE(Advanced) 2019, Paper-1,(4, –1)/62]**

Sol. $I = \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4} \frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ ….(1)

by a + b – x property गुणधर्म से

$I = \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4} \frac{dx}{(1+e^{-\sin x})(2-\cos 2x)} = \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4} \frac{e^{\sin x}dx}{(1+e^{\sin x})(2-\cos 2x)}dx$ …..(2)

adding (1) and और (2) जोड़ने पर

$2I = \frac{2}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4} \frac{(1+e^{\sin x})}{(1+e^{\sin x})(2-\cos 2x)}dx \Rightarrow \quad I = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4} \frac{1}{2-(2\cos^2 x-1)}dx = \frac{1}{\pi}\int_{-\frac{\pi}{4}}^{\pi/4} \frac{\sec^2 x}{3\sec^2 x - 2}dx$

put tanx = t, रखने पर $\sec^2 x dx = dt$

$= \frac{2}{\pi}\int_0^1 \frac{dt}{3t^2+1} = \frac{2}{3\pi}\frac{1}{\left(\frac{1}{\sqrt{3}}\right)}\left(\tan^{-1}\left(\frac{t}{1/\sqrt{3}}\right)\right)_0^1 = \frac{2}{\sqrt{3}\,\pi}\left(\tan^{-1}\left(\sqrt{3}\right) - \tan^{-1}(0)\right) = \frac{2}{\sqrt{3}\,\pi}\left(\frac{\pi}{3}\right) = \frac{2}{3\sqrt{3}}$

Now अब $27I^2 = 27 \times \frac{4}{27} = 4$

50. Let f : R → R be given by f(x) = (x – 1)(x – 2)(x – 5). Define $F(x) = \int_0^x f(t)\,dt,\ x > 0$. Then which of the following options is/are correct ? **[Definite integration_T]** **{[DI-DF]-T-305}**

(A) F(x) has a local maximum at x = 2

(B) F(x) has a local minimum at x = 1

(C) F(x) has two local maxima and one local minimum in (0, ∞)

(D) F(x) ≠ 0, for all x ∈ (0, 5) **[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62]**

माना कि $f : R \to R$, $f(x) = (x - 1)(x - 2)(x - 5)$ द्वारा दिया गया है। परिभाषित करें $F(x) = \int_0^x f(t)\,dt,\ x > 0$.

तब निम्न में से कौन सा (से) विकल्प सही है(हैं) ?

(A) F का एक स्थानीय उच्चतम (local maximum) x = 2 पर है।

(B) F का एक स्थानीय निम्नतम (local minimum) x = 1 पर है।

(C) F के दो स्थानीय उच्चतम और एक स्थानीय निम्नतम, (0, ∞) मे हैं।

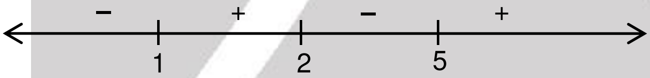
(D) सभी x ∈ (0, 5) के लिए F(x) ≠ 0 है।

Ans. **(ABD)**

Sol. $f(x) = (x - 1)\,(x - 2)\,(x - 5)$

$F(x) = \int_0^x f(t)dt$ as F(1) is negative and F(2) is also negative so F(x) cannot be zero for x ∈ (0, 5). So option (A) is correct.

$\therefore F'(x) = f(x) = (x - 1)\,(x - 2)\,(x - 5)$



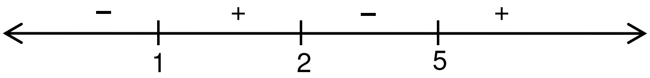
⇒ F(x) has two local minima points at x = 1 and x = 5

F(x) has one local maxima point at x = 2

Hindi. $f(x) = (x - 1)\,(x - 2)\,(x - 5)$

$F(x) = \int_0^x f(t)dt$ क्योंकि F(1) ऋणात्मक है तथा F(2) भी ऋणात्मक है इसलिए F(x) , x ∈ (0, 5) के लिये शून्य नहीं हो सकता

$\therefore F'(x) = f(x) = (x - 1)\,(x - 2)\,(x - 5)$



⇒ F(x), x = 1 और x = 5 दो स्थानीय निम्निष्ठ रखता है।

F(x) , x = 2 पर केवल एक उच्चिष्ठ रखता है।

51. The value of the integral **{[DI-SP]-M-302}**

$$\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{(\sqrt{\cos\theta}+\sqrt{\sin\theta})^5}\, d\theta$$

equals

समाकलन $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{(\sqrt{\sin\theta}+\sqrt{\cos\theta})^5}\, d\theta$ का मान है–

Ans. **(0.50)** **[JEE(Advanced) 2019, Paper-2 ,(4, –1)/62] [Definite integration_M]**

Sol. $I = 3\int_0^{\pi/2} \frac{\sqrt{\cos\theta}\, d\theta}{\left(\sqrt{\sin\theta}+\sqrt{\cos\theta}\right)^5} = 3\int_0^{\pi/2} \frac{3\sqrt{\sin\theta}\, d\theta}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}$

$\Rightarrow \int_0^{\pi/2} \frac{3\sqrt{\cos\theta}\, d\theta}{(\sqrt{\sin\theta}+\sqrt{\cos\theta})^5} = 3\int_0^{\pi/2} \frac{\sqrt{\sin\theta}\, d\theta}{(\sqrt{\cos\theta}+\sqrt{\sin\theta})^5} \Rightarrow 2I = 3\int_0^{\pi/2} \frac{d\theta}{(\sqrt{\sin\theta}+\sqrt{\cos\theta})^4}$

$\frac{2I}{3} = \int_0^{\pi/2} \frac{\sec^2\theta d\theta}{(\sqrt{\tan\theta}+1)^4}$

let $\tan\theta = t^2 \Rightarrow \sec^2\theta d\theta = 2tdt$

$\Rightarrow \frac{2I}{3} = \int_0^{\infty} \frac{2tdt}{(t+1)^4} \Rightarrow \frac{I}{3} = \int_0^{\infty}\left[\frac{1}{(t+1)^3} - \frac{1}{(t+1)^4}\right]dt$

$\Rightarrow I = \left|\frac{-3}{2(t+1)^2} + \frac{1}{(t+1)^3}\right|_0^{\infty} = \frac{3}{2} - 1 = \frac{1}{2}$

52. For $a \in R$, $|a| > 1$, let

$$\lim_{n\to\infty} \frac{1+\sqrt[3]{2}+\ldots\ldots\ldots+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(na+1)^2}+\frac{1}{(na+2)^2}+\ldots\ldots+\frac{1}{(na+n)^2}\right)} = 54.$$ Then possible value(s) a is/are –

[Definite integration_M] [JEE(Advanced) 2019, Paper-2 ,(4, –1)/62] {[DI-LS]-M-305}

माना कि $a \in R$, $|a| > 1$ के लिए

$$\lim_{n\to\infty} \frac{1+\sqrt[3]{2}+\ldots\ldots\ldots+\sqrt[3]{n}}{n^{7/3}\left(\frac{1}{(na+1)^2}+\frac{1}{(na+2)^2}+\ldots\ldots+\frac{1}{(na+n)^2}\right)} = 54,$$ तब a का (के) संभावित मान है (हैं)–

(A) 8 (B) –9 (C) 7 (D) –6

Ans. **(AB)**

Sol. $$\lim_{n\to\infty}\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\frac{1}{n}\left(\frac{n^2}{(na+1)^2}+\frac{n^2}{(na+2)^2}\cdots\cdots\frac{1}{(na+1)^2}\right)} =$$

$$\frac{\int_0^1 x^{1/3}dx}{\int_0^1 \frac{dx}{(a+x)^2}} = \frac{\left[\frac{3}{4}x^{4/3}\right]_0^1}{\left[-\frac{1}{a+x}\right]_0^1} = \frac{\frac{3}{4}}{\frac{1}{a}-\frac{1}{a+1}} = 54$$

$$\Rightarrow \frac{1}{a}-\frac{1}{a+1} = \frac{3}{4\times 54} \qquad\qquad \Rightarrow \frac{1}{a(a+1)} = \frac{1}{72}$$

a = 8 or a = – 9

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

भाग - II : JEE (MAIN) / AIEEE (पिछले वर्षों) के प्रश्न

1. Let p(x) be a function defined on **R** such that $p'(x) = p'(1 - x)$, for all $x \in [0, 1]$, p(0) = 1 and p(1) = 41. Then $\int_0^1 p(x)dx$ equals **[AIEEE 2010 (8, –2), 144]**

माना p(x), **R** पर परिभाषित एक ऐसा फलन है कि $p'(x) = p'(1 - x)$, प्रत्येक $x \in [0, 1]$ के लिए, p(0) = 1 तथा p(1) = 41 है। तो $\int_0^1 p(x)dx$ का मान है– **[AIEEE 2010 (8, –2), 144]**

(1*) 21 (2) 41 (3) 42 (4) $\sqrt{41}$

Ans.

Sol. $p'(x) = p'(1 - x)$

$\Rightarrow p(x) = -p(1 - x) + c$

put x = 0 रखने पर

p(0) = – p(1) + c

$\Rightarrow$ c = 42

$$I = \int_0^1 p(x)dx$$

$$I = \int_0^1 p(1-x)dx$$

$$2I = \int_0^1 (p(x)+p(1-x))\ dx = \int_0^1 c dx = \int_0^1 42 dx$$

2I = 42 $\Rightarrow$ I = 21

Hence correct option is (1)

अतः सही विकल्प (1) है।

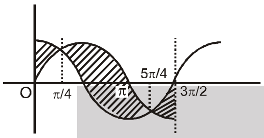
2. The area bounded by the curves y = cos x and y = sinx between the ordinates x = 0 and x = $\frac{3\pi}{2}$ is

[AIEEE 2010 (4, –1), 144]

कोटियों x = 0 तथा x = $\frac{3\pi}{2}$ के बीच वक्रों y = cos x तथा y = sinx से घिरे क्षेत्र का क्षेत्रफल है–
[AIEEE 2010 (4, –1), 144]
(1) $4\sqrt{2}+2$ (2) $4\sqrt{2}-1$ (3) $4\sqrt{2}+1$ (4*) $4\sqrt{2}-2$

Ans. **(4)**

Sol. Required area अभीष्ट क्षेत्रफल = $\int_{0}^{\pi/4}(\cos x-\sin x)dx + \int_{\pi/4}^{5\pi/4}(\sin x-\cos x)dx + \int_{5\pi/4}^{3\pi/2}(\cos x-\sin x)dx$



$= 2\left[\sin x+\cos x\right]_{0}^{\pi/4} + \left[-\cos x-\sin x\right]_{\pi/4}^{5\pi/4} = 4\sqrt{2}-2$

Hence correct option is (4)

अतः सही विकल्प (4) है।

3. For x ∈ $\left(0, \frac{5\pi}{2}\right)$, define f(x) = $\int_{0}^{x}\sqrt{t}\ \sin t\,dt$. Then f has : **[AIEEE 2011, I, (4, –1), 120]**

(1) local maximum at π and 2π.
(2) local minimum at π and 2π
(3) local minimum at π and local maximum at 2π.
(4*) local maximum at π and local minimum at 2π.

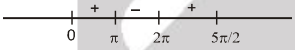
x ∈ $\left(0, \frac{5\pi}{2}\right)$ के लिए f(x) = $\int_{0}^{x}\sqrt{t}\ \sin t\,dt$ को परिभाषित कीजिए। तो f का :

(1) स्थानीय उच्चतम मान π तथा 2π पर है।
(2) स्थानीय निम्नतम मान π तथा 2π पर है।
(3) स्थानीय निम्नतम मान π पर तथा स्थानीय उच्चतम मान 2π पर है।
(4*) स्थानीय उच्चतम मान π पर तथा स्थानीय निम्नतम मान 2π पर है।

Sol. **(4)**

f(x) = $\int_{0}^{x}\sqrt{t}\sin t dt$

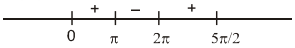
f′(x) = $\sqrt{x}$ sin x



local maximum at π
and local minimum at 2π **Ans.**

Hindi f(x) = $\int_{0}^{x}\sqrt{t}\sin t dt$

f′(x) = $\sqrt{x}$ sin x



π पर स्थानीय उच्चिष्ठ
तथा 2π पर स्थानीय निम्निष्ठ **Ans.**

4. Let [.] denote the greatest integer function then the value of $\int_{0}^{1.5} x\ \ [x^2]\,dx$ is : . **[AIEEE 2011, II, (4, –1), 120]**

माना [.] एक महत्तम पूर्णांक फलन दर्शाता है, तो $\int_{0}^{1.5} x\ \ [x^2]\ dx$ का मान है :

(1) 0 (2) $\frac{3}{2}$ (3*) $\frac{3}{4}$ (4) $\frac{5}{4}$

Sol. **(3)**

$\int_0^1 x\ [x^2]\ dx + \int_1^{\sqrt{2}} x\ [x^2]\ dx + \int_{\sqrt{2}}^{1.5} x\ [x^2]\ dx$

$\int_0^1 x.\ 0\ dx + \int_1^{\sqrt{2}} x\ dx + \int_{\sqrt{2}}^{1.5} 2x\ dx$

$0 + \left[\frac{x^2}{2}\right]_1^{\sqrt{2}} + \left[x^2\right]_{\sqrt{2}}^{1.5}$

$\frac{1}{2}(2-1) + (2.25-2)$

$\frac{1}{2} + .25$

$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

5. The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive x-axis is **[AIEEE 2011, I, (4, –1), 120]**

(1) $\frac{1}{2}$ square units (2) 1 square units (3*) $\frac{3}{2}$ square units (4) $\frac{5}{2}$ square units

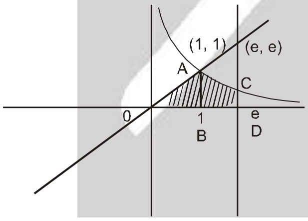
वक्रों $y = x$, $x = e$, $y = \frac{1}{x}$ तथा x-अक्ष को धन दिशा के बीच घिरे क्षेत्र का क्षेत्रफल है :**[AIEEE 2011, I, (4, –1), 120]**

(1) $\frac{1}{2}$ वर्ग इकाई (2) 1 वर्ग इकाई (3*) $\frac{3}{2}$ वर्ग इकाई (4) $\frac{5}{2}$ वर्ग इकाई

Sol. **(3)**

Required area

= OAB + ACDB

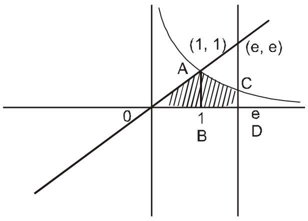


$= \frac{1}{2} \times 1 \times 1 + \int_1^e \frac{1}{x} dx$

$= \frac{1}{2} + \ (\ell nx)_1^e$

$= \frac{3}{2}$ square unit **Ans.**

Hindi अभिष्ठ क्षेत्रफल



$= \text{OAB} + \text{ACDB}$

$= \frac{1}{2}\times1\times1+\int_1^e \frac{1}{x}dx$

$= \frac{1}{2} + \ (\ell nx)_1^e$

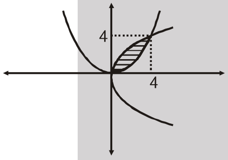
$= \frac{3}{2}$ वर्ग इकाई **Ans.**

6. The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$ is : **[AIEEE 2011, II, (4, –1), 120]**

वक्रों $y^2 = 4x$ तथा $x^2 = 4y$ द्वारा परिबद्ध क्षेत्र का क्षेत्रफल है :

(1) $\frac{32}{3}$ (2*) $\frac{16}{3}$ (3) $\frac{8}{3}$ (4) 0

Sol. **(2)**



Area क्षेत्रफल $= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right)dx$

$= \left(2\left(\frac{x^{3/2}}{3/2}\right) - \frac{x^3}{12}\right)_0^4$

$= \frac{4}{3}\times8 - \frac{64}{12}$

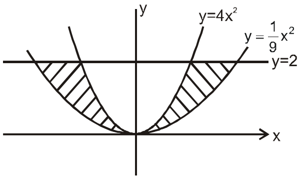
$= \frac{32}{3} - \frac{16}{3}$

$= \frac{16}{3}$

7. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$ is :
[AIEEE-2012, (4, –1)/120]

परवलयों $x^2 = \frac{y}{4}$ तथा $x^2 = 9y$ और रेखा $y = 2$ के बीच घिरे क्षेत्र का क्षेत्रफल है : **[AIEEE-2012, (4, –1)/120]**

(1) $20\sqrt{2}$ (2) $\frac{10\sqrt{2}}{3}$ (3*) $\frac{20\sqrt{2}}{3}$ (4) $10\sqrt{2}$



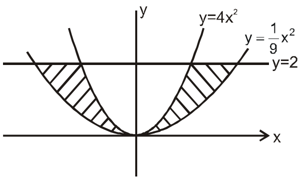
Sol.

$y = 4x^2$

$y = \frac{1}{9}\ x^2$

Area $= 2\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right)dy$

$$= 2\left[\frac{5}{2}\,\frac{y\sqrt{y}}{3/2}\right]_0^2 = 2.\ \frac{5}{3}\ 2\sqrt{2} = \frac{20\sqrt{2}}{3}$$



Hindi

$y = 4x^2$

$y = \frac{1}{9}\ x^2$

क्षेत्रफल $= 2\int_0^2\left(3\sqrt{y}-\frac{\sqrt{y}}{2}\right)dy = 2\left[\frac{5}{2}\,\frac{y\sqrt{y}}{3/2}\right]_0^2 = 2.\ 2\sqrt{2} = \frac{20\sqrt{2}}{3}$

8.* If $g(x) = \int_0^x \cos 4t\ \ dt$, then $g(x + \pi)$ equals **[AIEEE-2012, (4, –1)/120]**

यदि $g(x) = \int_0^x \cos 4t\ \ dt$ है, तो $g(x + \pi)$ बराबर है :

(1) $\frac{g(x)}{g(\pi)}$ (2*) $g(x) + g(\pi)$ (3*) $g(x) - g(\pi)$ (4) $g(x)\ .\ g(\pi)$

Sol. $g(x + \pi) = \int_0^{x+\pi} \cos 4t\ \ dt = g(x) + \int_0^{\pi}\cos 4t\ \ dt$

$\because \int_x^{\pi+x} \cos 4t\ \ dt = \int_x^{\pi}\cos 4t\ \ dt$ (as period of cos 4t is π)

$= g(x) + g(\pi)$

Here $g(\pi) = \int_0^{\pi}\cos 4t\ \ dt = 0$

so answers are (2) or (3)

Hindi $g(x + \pi) = \int_0^{x+\pi} \cos 4t\ \ dt = g(x) + \int_0^{\pi}\cos 4t\ \ dt$

$\because \int_x^{\pi+x} \cos 4t\ \ dt = \int_x^{\pi}\cos 4t\ \ dt$ (क्योंकि cos 4t का आवर्तकाल π है)

$= g(x) + g(\pi)$

यहाँ $g(\pi) = \int_0^{\pi}\cos 4t\ \ dt = 0$

इसलिए उत्तर (2) या (3) है।

9. Statement-I : The value of the integral $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$ is equal to $\pi/6$. **[AIEEE - 2013, (4, –1),360]**

Statement-II : . $\int_a^b f(x)dx = \int_a^b f(a+b-x)\ \ dx$ **[AIEEE - 2013, (4, –1),360]**

(1) Statement-I is true; Statement-II is true; Statement-II is a **correct** explanation for Statement-I.
(2) Statement-I is true; Statement-II is true; Statement-II is **not** a correct explanation for Statement-I.
(3) Statement-I is true; Statement-II is false.
(4*) Statement-I is false; Statement-II is true.

कथन-I : समाकलन $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ का मान π/6 है।

कथन-II :. $\int_a^b f(x)dx = \int_a^b f(a+b-x)\ \ dx$

(1) कथन-I सत्य है; कथन-II सत्य है; कथन-II कथन-I की **सही** व्याख्या है।

(2) कथन-I सत्य है; कथन-II सत्य है; कथन-II कथन-I की सही व्याख्या **नहीं** है।

(3) कथन-I सत्य है; कथन-II असत्य है।

(4) कथन-I असत्य है; कथन-II सत्य है।

Sol. (4)

Sol. (4)

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\ \ dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\ \ dx}{1+\sqrt{\tan x}}$$

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} dx$$

$$\Rightarrow \quad I = \frac{1}{2}\ ,\ \left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{\pi}{12}$$ statement -1 is false

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ it is property

Hindi (4)

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\ \ dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\ \ dx}{1+\sqrt{\tan x}}$$

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} dx$$

$$\Rightarrow \quad I = \frac{1}{2}\ ,\ \left[\frac{\pi}{3}-\frac{\pi}{6}\right] = \frac{\pi}{12}$$ कथन -1 असत्य है।

$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ गुणधर्म है।

10. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis, and lying in the first quadrant is :

वक्रों $y = \sqrt{x}$, $2y - x + 3 = 0$ तथा x-अक्ष से घिरे उस क्षेत्र, जो प्रथम चतुर्थांश में स्थित है, का (वर्ग इकाई में) क्षेत्रफल है

(1*) 9 (2) 36 (3) 18 (4) $\frac{27}{4}$

[AIEEE - 2013, (4, –1),360]

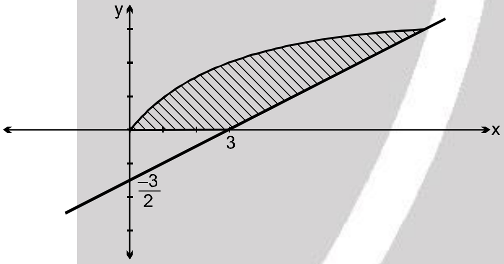
Sol. **(1)**

$y = \sqrt{x}$ ....(1)

and $2y - x + 3 = 0$ ....(2)

On solving both y = – 1, 3

Required area = $\int_0^3 \{(2y+3) \ -y^2\}\ dy$



$= \ y^2 + 3y - \frac{y^3}{3}\Big|_0^3$

$= 9 + 9 - 9$

$= 9.$

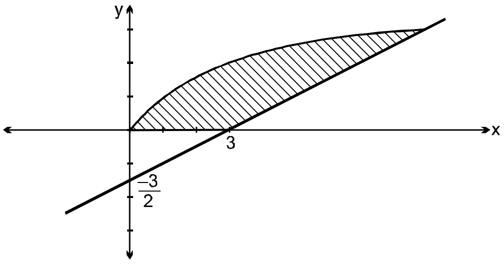
Hindi. **(1)**

$y = \sqrt{x}$ ....(1)

और $2y - x + 3 = 0$ ....(2)

हल करने पर y = – 1, 3

अभीष्ट क्षेत्रफल $= \int_0^3 \{(2y+3) \ -y^2\}\ dy$



$= \ y^2 + 3y - \frac{y^3}{3}\Big|_0^3$

$= 9 + 9 - 9$
$= 9.$

11. The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals :

[Definite Integration & Area] [JEE(Main) 2014, (4, – 1), 120]

समाकल $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ बराबर है :

[Definite Integration & Area] [JEE(Main) 2014, (4, – 1), 120]

(1) $4\sqrt{3}-4$ (2*) $4\sqrt{3}-4-\frac{\pi}{3}$ (3) $\pi - 4$ (4) $\frac{2\pi}{3}-4-4\sqrt{3}$

Sol. **Ans. (2)**

$$I = \int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx = \int_0^{\pi}|1-2\sin x/2|\ dx$$

$$= \int_0^{\pi/3}|1-2\sin x/2| + \int_{\pi/3}^{\pi}|1-2\sin x/2|\,dx = \int_0^{\pi/3}(1-2\sin x/2)dx + \int_{\pi/3}^{\pi}(2\sin x/2-1)dx$$

$$= \left(x+2\frac{\cos\frac{x}{2}}{\frac{1}{2}}\right)_0^{\pi/3} + \left(-2\frac{\cos\frac{x}{2}}{\frac{1}{2}}-x\right)_{\pi/3}^{\pi} = \left(\frac{\pi}{3}+4\frac{\sqrt{3}}{2}\right) - (4) + (0-\pi) - \left(\pi - 4\times\frac{\sqrt{3}}{2}-\frac{\pi}{3}\right)$$

$$= \frac{\pi}{3}+2\sqrt{3}-4-\pi+2\sqrt{3}+\pi/3 = -4 - \pi/3 + 4\sqrt{3}$$

12. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is :

[Definite Integration & Area] [JEE(Main) 2014, (4, – 1), 120]

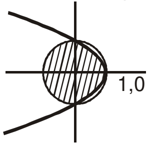
$A = \{(x, y) : x^2 + y^2 \le 1$ तथा $y^2 \le 1 - x\}$ के द्वारा प्रदत्त क्षेत्र का क्षेत्रफल है :

[Definite Integration & Area] [JEE(Main) 2014, (4, – 1), 120]

(1) $\frac{\pi}{2}-\frac{2}{3}$ (2) $\frac{\pi}{2}+\frac{2}{3}$ (3) $\frac{\pi}{2}+\frac{4}{3}$ (4) $\frac{\pi}{2}-\frac{4}{3}$

Sol. **Ans. (3)**

Intersection of $x^2 + y^2 = 1$ & $y^2 = 1 - x$
is $x = 0, 1$
The required portion is shaded as shown.
Area of region is area of semi-circle plus area bounded by parabola & y-axis.



Area of semi-circle is . $\frac{\pi}{2}$

Area bounded by parabola = $\frac{2}{3}$ of corresponding rectangle

$= \frac{2}{3} \times 1 \times 2 = \frac{4}{3}$

Hence total area $= \frac{\pi}{2} + \frac{4}{3}$.

Method - 1

Required area = area of semi circle + area bounded by parabola

$$= \frac{\pi}{2} + \int_0^1 (1-y^2)\ dy = \frac{\pi}{2} + 2\left(y - \frac{y^3}{3}\right)_0^1$$

$$= \frac{\pi}{2} + 2\left(1 - \frac{1}{3}\right) \Rightarrow \frac{\pi}{2} + \frac{4}{3}$$

Hindi.

$x^2 + y^2 = 1$ & $y^2 = 1 - x$ के प्रतिच्छेदन से

$x = 0, 1$

अभिष्ट भाग चित्र मे दर्शाया गया है

क्षेत्र का क्षेत्रफल, अर्द्धवृत्त का क्षेत्रफल और y-अक्ष तथा परवलय से परिबद्ध क्षेत्र का योगफल है

अर्द्ध वृत्त का क्षेत्रफल $\frac{\pi}{2}$.

परवलय से परिबद्ध क्षेत्रफल = $\frac{2}{3}$ संगत आयत का क्षेत्रफल

$$= \frac{2}{3} \times 1 \times 2 = \frac{4}{3}$$

अतः कुल क्षेत्रफल = $\frac{\pi}{2} + \frac{4}{3}$. .

Method - 1

अभिष्ट क्षेत्रफल = अर्द्धवृत्त का क्षेत्रफल + परवलय से परिबद्ध क्षेत्रफल

$$= \frac{\pi}{2} + \int_0^1 (1-y^2)\ dy = \frac{\pi}{2} + 2\left(y - \frac{y^3}{3}\right)_0^1$$

$$= \frac{\pi}{2} + 2\left(1 - \frac{1}{3}\right) \Rightarrow \frac{\pi}{2} + \frac{4}{3}$$

13. The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$ is equal to **[Definite integration]**

[JEE(Main) 2015, (4, – 1), 120]

(1) 2 (2) 4 (3) 1 (4) 6

समाकल $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36 - 12x + x^2)} dx$ बराबर है– **[JEE(Main) 2015, (4, – 1), 120]**

(1) 2 (2) 4 (3) 1 (4) 6

Ans. **(3)**

Sol. $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log\left(x^2 - 12x + 36\right)} dx$

$$I = \frac{\text{२}}{\text{२}} \int_2^4 \frac{\log x}{\log x + \log(6-x)} dx \qquad \text{...(i)}$$

$$I = \int_2^4 \frac{\log(6-x)}{\log(6-x) + \log x} dx \qquad \left\{\int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right\} \qquad \text{...(ii)}$$

Equation (i) & (ii) gives

$$2I = \int_2^4 \frac{\log x + \log(6-x)}{\log x + \log(6-x)} dx = \int_2^4 dx = 2$$

Hence I = 1

Hindi $I = \int_2^4 \frac{\log x^2}{\log x^2 + \log(x^2 - 12x + 36)} dx$

$I = \frac{२}{२}\int_2^4 \frac{\log x}{\log x + \log(6 - x)} dx$ ...(i)

$I = \int_2^4 \frac{\log(6 - x)}{\log(6 - x) + \log x} dx$ $\left\{\int_a^b f(x)dx = \int_a^b f(a + b - x)dx\right\}$ ...(ii)

समीकरण (i) और (ii) से $2I = \int_2^4 \frac{\log x + \log(6 - x)}{\log x + \log(6 - x)} dx = \int_2^4 dx = 2$

अतः $I = 1$

14. The area (in sq. units) of the region described by {(x, y); $y^2 \le 2x$ and $y \ge 4x - 1$} is **[JEE(Main) 2015, (4, – 1), 120]**

(1) $\frac{7}{32}$ (2) $\frac{5}{64}$ (3) $\frac{15}{64}$ (4) $\frac{9}{32}$

{(x, y); $y^2 \le 2x$ तथा $y \ge 4x - 1$} द्वारा परिभाषित क्षेत्र का क्षेत्रफल (वर्ग इकाईयों) में है–

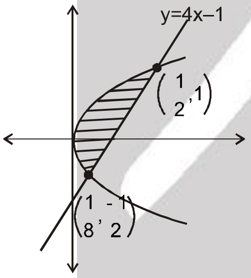
[Definite integration and its application] [JEE(Main) 2015, (4, – 1), 120]

(1) $\frac{7}{32}$ (2) $\frac{5}{64}$ (3) $\frac{15}{64}$ (4) $\frac{9}{32}$

Ans. **(4)**

Sol. $\int_{-1/2}^{2} \left(\frac{y+1}{4} - \frac{y^2}{2}\right) dy$

$\Rightarrow \frac{1}{4}\left\{\frac{y^2}{2} + y\right\}_{-1/2}^{1} - \frac{1}{6}\{y^3\}_{-1/2}^{1}$



$\Rightarrow \frac{1}{4}\left\{\left(\frac{1}{2} + 1\right) - \left(\frac{1}{8} - \frac{1}{2}\right)\right\} - \frac{1}{6}\left\{1 + \frac{1}{8}\right\}$

$\Rightarrow \frac{1}{4}\left\{\frac{3}{2} + \frac{3}{8}\right\} - \frac{1}{6}\left\{\frac{9}{8}\right\}$ $\Rightarrow \frac{15}{32} - \frac{6}{32} = \frac{9}{32}$

15. The area (in sq.units) of the region {(x,y) : $y^2 \ge 2x$ and $x^2 + y^2 \le 4x$, $x \ge 0$, $y \ge 0$} is

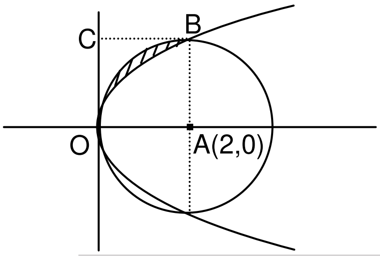
क्षेत्र {(x,y) : $y^2 \ge 2x$ तथा $x^2 + y^2 \le 4x$, $x \ge 0$, $y \ge 0$} का क्षेत्रफल (वर्ग इकाईयों में) है– **[JEE Main 2016]**

(1) $\pi - \frac{8}{3}$ (2) $\pi - \frac{4\sqrt{2}}{3}$ (3) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ (4) $\pi - \frac{4}{3}$

Ans. **(1)**

Sol. $y^2 = 2x$ and $x^2 + y^2 = 4x$ meet at O(0, 0) and B(2, 2) {(2, –2) is not considered as x, y ≥ 0}

$y^2 = 2x$ और $x^2 + y^2 = 4x$, O(0, 0) पर मिलते है और B(2, 2) {(2, –2) ये नहीं लिया गया है x, y ≥ 0}



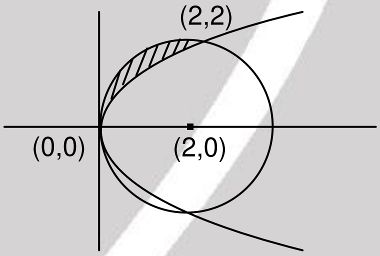
Now required area $= \text{(Area of quadrant of circle)} - \frac{2}{3}\text{(Area of rectangle OABC)}$

अभीष्ट क्षैत्रफल = (वृत्त का चतुर्थांश का क्षैत्रफल) – $\frac{2}{3}$ (आयत OABC का क्षैत्रफल)

$$= \pi - \frac{2}{3}.(2.2) = \pi - \frac{8}{3}$$

Alter :

$y^2 \geq 2x$ & $x^2 + y^2 \leq 4x$; $x \geq 0, y \geq 0$



$x^2 + 2x = 4x$

$x^2 - 2x = 0$

$x = 0, 2$

$$\int_0^2 (\sqrt{4x - x^2} - \sqrt{2x})\, dx \;=\; \pi - \frac{8}{3}$$

$$\int_0^2 (\sqrt{4 - (x-2)^2} - \sqrt{2}\sqrt{x})\, dx$$

$$\left(\left(\frac{(x-2)}{2}\sqrt{4x - x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right) - \frac{\sqrt{2}(x^{3/2})}{3}(2)\right)_0^2$$

$$\left(\frac{-2\sqrt{2}}{3}(2^{3/2}) - \left(2\sin^{-1}(-1)\right)\right)$$

$$\frac{-2\sqrt{2}}{3}(2\sqrt{2})-2\left(\frac{-\pi}{2}\right) = \pi-\frac{8}{3}$$

16. $\lim\limits_{n\to\infty}\left(\frac{(n+1)(n+2).......3n}{n^{2n}}\right)^{1/n}$ is equal to : **[JEE Main 2016]**

$\lim\limits_{n\to\infty}\left(\frac{(n+1)(n+2).......3n}{n^{2n}}\right)^{1/n}$ बराबर है :

(1) $\frac{27}{e^2}$ (2) $\frac{9}{e^2}$ (3) 3 log3 – 2 (4) $\frac{18}{e^4}$

Ans. **(1)**

Sol. $p = \lim\limits_{n\to\infty}\left(\frac{(n+1)(n+2)......(n+2n)}{n^{2n}}\right)$

$$\log p = \frac{1}{n}\left(\lim_{n\to\infty}\sum_{r=1}^{2n}\log\left(1+\frac{r}{n}\right)\right)$$

$$\log p = \int_0^2 \log(1+x)dx$$

$$\log p = (x\log(1+x))_0^2 - \int_0^2 \frac{x}{1+x}dx$$

$$\log p = 2\log 3 - \int_0^2\left(1-\frac{1}{1+x}\right)dx$$

$$\log p = 2\log 3 - (x - \log(1+x))_0^2$$

$$\log p = 2\log 3 - (2-\log 3)$$

$$\log p = 3\log 3 - 2 = \log\frac{27}{e^2}$$

$$p = \frac{27}{e^2}$$

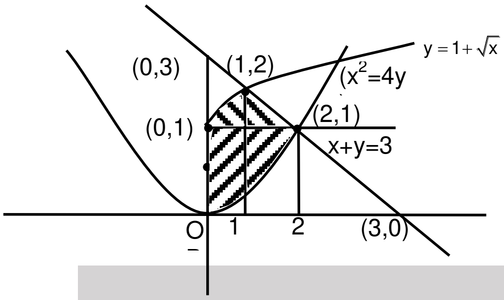
17. The area (in sq. units) of the region $\{(x, y) : x \geq 0,\ x + y \leq 3,\ x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is :

क्षेत्र $\{(x, y) : x \geq 0,\ x + y \leq 3,\ x^2 \leq 4y$ तथा $y \leq 1 + \sqrt{x}\}$ का क्षेत्रफल (वर्ग इकाइयों) में है:

[JEE(Main) 2017, (4, – 1), 120]

(1) $\frac{59}{12}$ (2) $\frac{3}{2}$ (3) $\frac{7}{3}$ (4) $\frac{5}{2}$

Ans. (4)



Sol.

$y = 1 + \sqrt{x}$

$(y - 1)^2 \le x$

Required area अभीष्ट क्षेत्रफल $= \int_0^1 \left(1+\sqrt{x}\right)dx + \int_1^2 (3-x)dx - \int_0^2 \frac{x^2}{4}dx$

$= \left(x + \frac{2x^{3/2}}{3}\right)\Bigg|_0^1 + \left(3x - \frac{x^2}{7}\right)\Bigg|_1^2 - \left(\frac{x^3}{12}\right)_0^2 = 1 + \frac{2}{3} + \left\{(6-2) - \frac{5}{2}\right\} - \frac{8}{12}$

$= 1 + \frac{2}{3} + \left(4 - \frac{5}{2}\right) - \frac{2}{3} = 1 + \frac{3}{2} = \frac{5}{2}$

18. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to **[JEE(Main) 2017, (4, – 1), 120]**

समाकल $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ बराबर है– **[JEE(Main) 2017, (4, – 1), 120]**

(1) –2 (2) 2 (3) 4 (4) –1

Ans. **(2)**

Sol. $I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1+\cos x}dx$

Using property गुणधर्म , $\int_a^b f(x).dx = \int_a^b f(a+b-x).dx$ का उपयोग करने पर

$$I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1-\cos x}dx$$

On adding we get, योग करने पर

$$2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{1}{1-\cos^2 x}dx$$

$$2I = \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2\text{cosec}^2x.dx$$

$$I = (-\cot x)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = 2$$

19. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}dx$ is : **[Definite Integration]**

$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x}dx$ का मान है : **[JEE(Main) 2018, (4, – 1), 120]**

(1) 4π (2*) $\frac{\pi}{4}$ (3) $\frac{\pi}{8}$ (4) $\frac{\pi}{2}$

Sol. **(2)**

$$I = \int_{+\pi/2}^{+\pi/2} \frac{\sin^2 x}{1+2^x}dx$$

$$I = \int_{0}^{+\pi/2} \left(\frac{\sin^2 x}{1+2^x} + \frac{\sin^2 x}{1+2^{-x}}\right)dx$$

property $\int_{-9}^{+9} f(x)dx = \int_0^9 (f(x)+(-x))dx$

$$I = \int_0^{\pi/2} \sin^2 x \quad dx = \int_0^{\pi/2} \frac{(1-\cos(2x))}{2}dx = \left(\frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right)_0^{\pi/2}\right) = \frac{1}{2}[\pi/2 - 0] = \pi/4$$

20. Let g(x) = cosx², f(x) = $\sqrt{x}$, and α, β (α < β) be the roots of the quadratic equation 18x² – 9πx + π² = 0. Then the area (in sq. units) bounded by the curve y = (gof) (x) and the lines x=α, x = β and y = 0, is

Definite Integration **[JEE(Main) 2018, (4, – 1), 120]**

माना $g(x) = \cos x^2$, $f(x) = \sqrt{x}$, तथा α, β $(\alpha < \beta)$ द्विघाती समीकरण $18x^2 - 9\pi x + \pi^2 = 0$ के मूल हैं। तो वक्र $y = (gof)(x)$ तथा रेखाओं $x=\alpha$, $x = \beta$ तथा $y = 0$ द्वारा घिरे क्षेत्र का क्षेत्रफल (वर्ग इकाइयों में) है :

(1) $\frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$ (2) $\frac{1}{2}\left(\sqrt{2}-1\right)$ (3*) $\frac{1}{2}\left(\sqrt{3}-1\right)$ (4) $\frac{1}{2}\left(\sqrt{3}+1\right)$

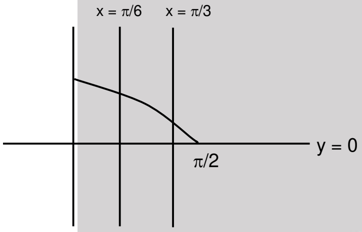
Sol. **(3)**

$g(x) = \cos(x^2)$, $f(x) = \sqrt{x}$

$18x^2 - 9\pi x + \pi^2 = 0 \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$ $(\alpha + \beta) = (6x - \pi)(3x - \pi) = 0 \Rightarrow x = \pi/6 , \pi/3$

$y = gof(x) = g(f(x)) = g\left(\sqrt{x}\right) = \cos(x)$

$\alpha = \pi/6$, $\beta = \pi/3$, $x = \pi/6$, $x = \pi/3$



$$A = \int_{\pi/6}^{\pi/3} \cos x dx = (\sin x)_{\pi/6}^{\pi/3} = \frac{\sqrt{3}}{2} - \frac{1}{2} = \frac{\sqrt{3}-1}{2}$$

21. Let $I = \int_a^b (x^4 - 2x^2)dx$. If I is minimum then the ordered pair (a, b) is :

माना $I = \int_a^b (x^4 - 2x^2)dx$ है। यदि I न्यूनतम है, तो क्रमित युग्म (a, b) है :

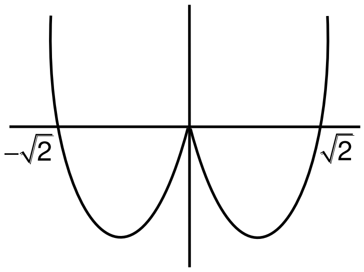
[JEE(Main) 2019, Online (10-01-19),P-1 (4, – 1), 120]

(1) $(0, \sqrt{2})$ (2) $(\sqrt{2}, -\sqrt{2})$ (3) $(-\sqrt{2}, \sqrt{2})$ (4) $(-\sqrt{2}, 0)$

Ans. **(3)**

Sol. In this question we are assuming b is greater than a otherwise no option is correct

$$\int_a^b \left(x^4 - 2x^2\right)dx$$



From figure min area is चित्र से न्यूनतम क्षेत्रफल $\left(-\sqrt{2},\sqrt{2}\right)$

22. The value of $\int_{-\pi/2}^{\pi/2}\frac{dx}{[x]+[\sin x]+4}$, where [t] denotes the greatest less than or equal to t, is :

$\int_{-\pi/2}^{\pi/2}\frac{dx}{[x]+[\sin x]+4}$ का मान, जहाँ [t] वह महत्तम पूर्णांक है जो t से कम या उसके बराबर है, है:

[JEE(Main) 2019, Online (10-01-19),P-2 (4, – 1), 120]

(1) $\frac{1}{12}(7\pi-5)$ (2) $\frac{3}{10}(4\pi-3)$ (3) $\frac{3}{20}(4\pi-3)$ (4) $\frac{1}{12}(7\pi+5)$

Ans. **(3)**

Sol. $\int_{-\frac{\pi}{2}}^{\pi/2}\frac{dx}{[x]+[\sin x]+4} = \int_{-\frac{\pi}{2}}^{0}\frac{dx}{[x]+3}+\int_{0}^{\pi/2}\frac{dx}{[x]+4} = \int_{-\frac{\pi}{2}}^{-1}\frac{dx}{1}+\int_{-1}^{0}\frac{dx}{2}+\int_{0}^{1}\frac{dx}{4}+\int_{1}^{\pi/2}\frac{dx}{5}$

$= [x]_{-\frac{\pi}{2}}^{-1}+\left[\frac{x}{2}\right]_{-1}^{0}+\left[\frac{x}{4}\right]_{0}^{1}+\left[\frac{x}{5}\right]_{1}^{\pi/2} = \left(-1+\frac{\pi}{2}\right)+\left(0+\frac{1}{2}\right)+\frac{1}{4}+\frac{\pi}{10}-\frac{1}{5}$

$= \frac{-20+10\pi+10+5+2\pi-4}{20} = \frac{12\pi-9}{20} = \frac{3}{20}(4\pi-3)$

23. The integral $\int_{1}^{e}\left\{\left(\frac{x}{e}\right)^{2x}-\left(\frac{e}{x}\right)^{x}\right\}\log_e x\,dx$ is equal to

समाकल $\int_{1}^{e}\left\{\left(\frac{x}{e}\right)^{2x}-\left(\frac{e}{x}\right)^{x}\right\}\log_e x\,dx$ बराबर है–**[JEE(Main) 2019, Online (12-01-19),P-2 (4, – 1), 120]**

(1) $\frac{3}{2}-\frac{1}{e}-\frac{1}{2e^2}$ (2) $\frac{3}{2}-e-\frac{1}{2e^2}$ (3) $\frac{1}{2}-e-\frac{1}{e^2}$ (4) $-\frac{1}{2}+\frac{1}{e}-\frac{1}{2e^2}$

Ans. (2)

Sol. Let माना $\left(\frac{x}{e}\right)^{x}=t \Rightarrow x(\ell nx-1)=\ell nt \Rightarrow (\ell nx)dx=\frac{dt}{t}$

$I = \int_{1/e}^{1}\left(t^2-\frac{1}{t}\right)\frac{dt}{t} = \int_{1/e}^{1}\left(t-\frac{1}{t^2}\right)dt = \left(\frac{t^2}{2}+\frac{1}{t}\right)_{1/e}^{1} = \left(\frac{1}{2}+1\right)-\left(\frac{1}{2e^2}+e\right) = \frac{3}{2}-\frac{1}{2e^2}-e$

24. Let f : R → R be a continuously differentiable function such that f(2) = 6 and f'(2) = $\frac{1}{48}$ .If $\int_{6}^{f(x)} 4t^3 dt = 9$ (x – 2) g(x), then $\lim\limits_{x \to 2} g(x)$ is equal to

माना f : R → R एक संततत : अवकलनीय फलन इस प्रकार है कि f(2) = 6 तथा f'(2) = $\frac{1}{48}$ यदि $\int_{6}^{f(x)} 4t^3 dt = 9$ (x – 2) g(x), तो $\lim\limits_{x \to 2} g(x)$ बराबर है :

(1) 18 (2) 14 (3) 12 (4) 36

Ans. (1) **[JEE(Main) 2019, Online (12-04-19),P-1 (4, – 1), 120] [Definite Integration]**

Sol. $\int_{6}^{f(x)} 4x^3 dx = g(x).(x-2) \Rightarrow g(x) = \frac{(f(x))^4 - 6^4}{x-2}$

$\therefore \lim\limits_{x \to 2} g(x) = \lim\limits_{x \to 2} \frac{(f(x))^4 - 6^4}{x-2} = \lim\limits_{x \to 2} \frac{4f^3(x).f'(x)}{1} = 4 \times 6^3 \times \frac{1}{48} = 18$

25. Let f(x) = $\int_{0}^{x} g(t)dt$, where g is a non-zero even function. If f(x + 5) = g (x), then $\int_{0}^{x} f(t)dt$ equals :

माना f(x) = $\int_{0}^{x} g(t)dt$ है, जहाँ g एक शून्येत्तर समफलन है। यदि f(x + 5) = g (x) है तो $\int_{0}^{x} f(t)dt$ बराबर है:

(1) $\int_{5}^{x+5} g(t)dt$ (2*) $\int_{x+5}^{5} g(t)dt$ (3) $5\int_{x+5}^{5} g(t)dt$ (4) $2\int_{5}^{x+5} g(t)dt$

Ans. (2) **[JEE(Main) 2019, Online (08-04-19),P-2 (4, – 1), 120] [Definite Integration]**

Sol. f(0) = 0 and g(x) is even, so f(x) is odd function.

g(x) = f(x + 5)

Replacing x by –x

g(– x) = f(–x + 5)

g(x) = –f(x – 5) {g is even and f is odd}

Replacing x by x + 5

f(x) = –g(x + 5)

Now I = $\int_{0}^{x} f(t)dt = -\int_{0}^{x} g(t+5)dt$

Put t + 5 = k

$$I = -\int_{5}^{x+5} g(k)\,dk = \int_{x+5}^{5} g(t)\,dt$$

26. $\lim\limits_{n\to\infty}\left(\dfrac{(n+1)^{1/3}}{n^{4/3}}+\dfrac{(n+2)^{1/3}}{n^{4/3}}+....+\dfrac{(2n)^{1/3}}{n^{4/3}}\right)$ is equal to :

$\lim\limits_{n\to\infty}\left(\dfrac{(n+1)^{1/3}}{n^{4/3}}+\dfrac{(n+2)^{1/3}}{n^{4/3}}+....+\dfrac{(2n)^{1/3}}{n^{4/3}}\right)$ बराबर है :

(1) $\frac{3}{4}(2)^{4/3}-\frac{3}{4}$ (2) $\frac{3}{4}(2)^{4/3}-\frac{4}{3}$ (3) $\frac{4}{3}(2)^{4/3}$ (4) $\frac{4}{3}(2)^{3/4}$

Ans. (1) [JEE(Main) 2019, Online (10-04-19),P-1 (4, – 1), 120] [Definite Integration]

Sol. $\lim\limits_{n\to\infty}\left\{\left(1+\frac{1}{n}\right)^{1/3}+\left(1+\frac{2}{n}\right)^{1/3}+.....+\left(1+\frac{n}{n}\right)^{1/3}\right\}\frac{1}{n}=\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\left(1+\frac{r}{n}\right)^{1/3}\frac{1}{n}$

$$=\int_0^1 (1+x)^{1/3}dx = \frac{1}{4/3}.(1+x)^{4/3}\Big|_0^1 = \frac{3}{4}(2^{4/3}-1)$$

27. The value of the integral $\int_0^1 x\cot^{-1}(1-x^2+x^4)dx$ is :

समाकल $\int_0^1 x\cot^{-1}(1-x^2+x^4)dx$ का मान है–

(1) $\frac{\pi}{2}-\frac{1}{2}\log_e 2$ (2) $\frac{\pi}{4}-\log_e 2$ (3) $\frac{\pi}{2}-\log_e 2$ (4) $\frac{\pi}{4}-\frac{1}{2}\log_e 2$

Ans. (2) [JEE(Main) 2019, Online (09-04-19),P-2 (4, – 1), 120] [Definite Integration]

Sol. $\int_0^1 x\tan^{-1}\left(\dfrac{1}{1-x^2+x^4}\right)dx$

$\Rightarrow$ Put $x^2 = t$

$$=\frac{1}{2}\int_0^1 \tan^{-1}\left(\frac{1}{1-t+t^2}\right)dt$$

$$=\frac{1}{2}\int_0^1 \tan^{-1}\left(\frac{t+(1+t)}{1-t(1-t)}\right)dt$$

$$= \frac{1}{2}\int_0^1 \left(\tan^{-1} t + \tan^{-1}(1-t)\right) dt$$

$$= \frac{1}{2}\int_0^1 \left(\tan^{-1}(1-t)\right)dt + \frac{1}{2}\int_0^1 \left(\tan^{-1}(1-t)\right)dt$$

$$= \int_0^1 \left(\tan^{-1}(1-t)\right)dt = \int_0^1 \left(\tan^{-1}(t)\right)dt$$

Put $\tan^{-1} t = k$

$$= \int_0^{\frac{\pi}{4}} k\sec^2 k dk = \frac{\pi}{4} - \frac{1}{2}\ell n2 \text{ (using by parts)}$$

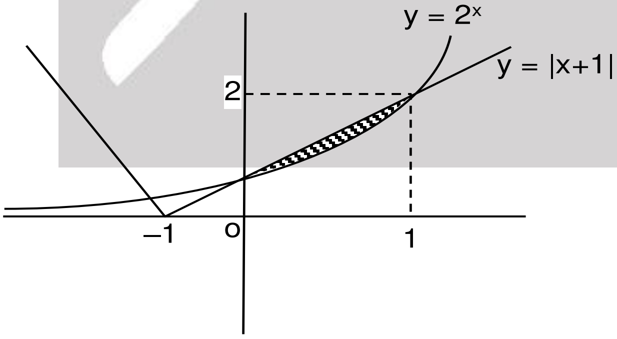
28. The area (in sq. units) of the region bounded by the curves $y = 2^x$ and $y = |x + 1|$, in the first quadrant is

वक्रों $y = 2^x$ तथा $y = |x + 1|$ द्वारा प्रथम चतुर्थांश में परिबद्ध क्षेत्र का क्षेत्रफल (वर्ग इकाइयों में) है–

(1) $\frac{3}{2} - \frac{1}{\log_e 2}$ (2) $\frac{3}{2}$ (3) $\frac{1}{2}$ (4) $\log_e 2 + \frac{3}{2}$

Ans. **(1)** **[JEE(Main) 2019, Online (10-04-19),P-2 (4, – 1), 120]** **[Definite Integration]**

Sol. Area $= \int_0^1 (|x+1| - 2^x)\, dx = \int_0^1 (x+1-2^x)\, dx = \left(\frac{x^2}{2} + x - \frac{2^x}{\ell n 2}\right)_0^1$



$$= \left(\frac{1}{2} + 1 - \frac{2}{\ell n 2}\right) - \left(0 + 0 - \frac{1}{\ell n2}\right) = \frac{3}{2} - \frac{1}{\ell n2}$$

29. If f(a + b + 1 – x) = f(x), for all x, where a and b are fixed positive real numbers, then $\frac{1}{(a+b)}\int_a^b x(f(x)+f(x+1))dx$ is equal to

यदि सभी x के लिए f(a + b + 1 – x) = f(x) है, जबकि a तथा b स्थिर (fixed) धन वास्तविक संख्याएँ हैं, तो $\frac{1}{(a+b)}\int_a^b x(f(x)+f(x+1))dx$ बराबर है :

(1) $\int_{a+1}^{b+1} f(x+1)dx$ (2) $\int_{a-1}^{b-1} f(x+1)dx$ (3) $\int_{a+1}^{b+1} f(x)dx$ (4) $\int_{a-1}^{b-1} f(x)dx$

Ans. **(2)** **[JEE(Main) 2020, Online (07-01-20),P-1 (4, – 1), 120]**

Note : Answer given by NTA is (3) which is wrong

Sol. $I = \frac{1}{(a+b)}\int_a^b x[f(x)+f(x+1)]dx$ ......(1)

$x \to a + b - x$

$I = \frac{1}{(a+b)}\int_a^b (a+b-x)[f(a+b-x)+f(a+b+1-x)]dx$

$I = \frac{1}{(a+b)}\int_a^b (a+b-x)[f(x+1)+f(x)]dx$ ....(2)

[∵ put x → x + 1 in given equation]

[∵दी गई समीकरण में x → x + 1 रखने पर]

(1) + (2)

$2I = \int_a^b [f(x+1)+f(x)]dx$

$2I = \int_a^b f(x+1)dx + \int_a^b f(x)dx$

$\int_a^b f(a+b+1-x)dx + \int_a^b f(x)dx$

$2I = 2\int_a^b f(x)dx$

$$I = \int_{a}^{b} f(x)\,dx$$

Put x = t + 1 $\Rightarrow$ $I = \int_{a-1}^{b-1} f(t+1)\,dt$

30. For a > 0, let the curves C_1: $y^2 = ax$ and C_2 :$x^2 = ay$ intersect at origin O and a point P. Let the line x = b (0 < b < a) intersect the chord OP and the x-axis at points Q and R, respectively. If the line x = b bisects the area bounded by the curves, C_1 and C_2, and the area of ΔOQR = $\frac{1}{2}$, then 'a' satisfies the equation :

a > 0 के लिए, माना वक्र C_1: $y^2 = ax$ तथा C_2 :$x^2 = ay$, मूलबिंदु O तथा एक बिंदु P पर काटते हैं। माना रेखा x = b (0 < b < a), जीवा OP तथा x-अक्ष को क्रमशः बिंदुओं Q तथा R पर काटती है। यदि रेखा x = b, वक्रों C_1 तथा C_2 द्वारा परिबद्ध क्षेत्र को समद्विभाजित करती है तथा ΔOQR = $\frac{1}{2}$ है, तो 'a' जिस समीकरण को संतुष्ट करता है, वह है:

(1) $x^6 - 12x^3 - 4 = 0$ (2) $x^6 - 6x^3 + 4 = 0$ (3) $x^6 - 12x^3 + 4 = 0$ (4) $x^6 + 6x^3 - 4 = 0$

Ans. (3) **[JEE(Main) 2020, Online (08-01-20),P-1 (4, –1), 120]**

Sol. $$\int_{0}^{b}\left(\sqrt{ax} - \frac{x^2}{a}\right)dx = \frac{a^2}{6}$$

$\Rightarrow$ $\frac{2}{3}\sqrt{a}\, b^{\frac{3}{2}} - \frac{b^3}{3a} = \frac{a^2}{6}$ ....(i)

also area of Δ OQR का क्षेत्रफल = $\frac{1}{2}$

$\frac{1}{2}b^2 = \frac{1}{2}$ $\Rightarrow$ b = 1

Put in (i) मे रखने पर

$\Rightarrow$ $4a\sqrt{a} - 2 = a^3$

$\Rightarrow$ $a^6 + 4a^3 + 4 = 16a^3$

$\Rightarrow$ $a^6 - 12a^3 + 4 = 0$

31. If I = $\int_{1}^{2}\frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, then

यदि I = $\int_{1}^{2}\frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, तब **[JEE(Main) 2020, Online (08-01-20),P-2 (4, –1), 120]**

(1) $\frac{1}{6} < I^2 < \frac{1}{2}$ (2) $\frac{1}{16} < I^2 < \frac{1}{9}$ (3) $\frac{1}{8} < I^2 < \frac{1}{4}$ (4) $\frac{1}{9} < I^2 < \frac{1}{8}$

Ans. **(4)**

Sol. $f(x) = \dfrac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$

$$f'(x) = \frac{-1}{2}\,\frac{(6x^2 - 18x + 12)}{(2x^3 - 9x^2 + 12x + 4)^{\frac{3}{2}}} = \frac{-6(x-1)(x-2)}{2(2x^3 - 9x^2 + 12x + 4)^{\frac{3}{2}}}$$

$f(1) = \dfrac{1}{3}$, $\quad f(2) = \dfrac{1}{\sqrt{8}}$

$\dfrac{1}{3} < I < \dfrac{1}{\sqrt{8}}$

32. Let a function f : [0, 5] → R be continuous, f(1) = 3 and F be defined as : $f(x) = \int_1^x t^2 g(t)dt$, where $g(t) = \int_1^t f(u)du$. Then for the function F, the point x = 1 is :

(1) a point of local minima

(2) a point of local maxima.

(3) not a critical point

(4) a point of inflection.

माना एक फलन f : [0, 5] → R संतत है, तथा, f(1) = 3 है तथा $F(x) = f(x) = \int_1^x t^2 g(t)dt$, द्वारा परिभाषित है, जहां $g(t) = \int_1^t f(u)du$. है, तो फलन F के लिए बिन्दु x = 1

(1) स्थानीय निम्ननिष्ठ बिन्दु है।

(2) स्थानीय उच्चिष्ठ बिन्दु है।

(3) क्रांतिक बिन्दु नहीं है।

(4) नति परिवर्तन (inflection) बिन्दु है।

Ans. **(1)** **[JEE(Main) 2020, Online (09-01-20),P-2 (4, –1), 120]**

Sol. $F'(x) = x^2 g(x)$

$\Rightarrow F'(1) = 1.g(1) = 0$ .... (1) $\quad (\because g(1) = 0)$

Now अब $F''(x) = 2xg(x) + x^2 g'(x)$

$\Rightarrow F''(x) = 2xg(x) + x^2 f(x)$ $\quad (\because g'(x) = f(x))$

$\Rightarrow F''(1) = 0 + 1 \times 3$

$\Rightarrow F''(1) = 3$ .... (2)

From (1) and (2) F(x) has local minimum at x = 1

(1) तथा (2) से F(x), x = 1 पर स्थानिय निम्निष्ठ है

High Level Problems (HLP)

1. Find the integral value of a for which $\int_0^{\frac{\pi}{2}} (\sin x + a\cos x)^3 dx - \frac{4a}{\pi - 2}\int_0^{\frac{\pi}{2}} x\cos x dx = 2$

a का पूर्णांक मान ज्ञात कीजिए जिसके लिए $\int_0^{\frac{\pi}{2}} (\sin x + a\cos x)^3 dx - \frac{4a}{\pi - 2}\int_0^{\frac{\pi}{2}} x\cos x dx = 2$

Ans. –1

Sol. $\int_0^{\pi/2} (\sin x + a\cos x)^3 dx - \frac{4a}{\pi - 2}\int_0^{\pi/2} x\cos x dx = 2$

$\Rightarrow \int_0^{\pi/2} (\sin^3 x + 3a\sin^2 \cos x + 3a^2 \sin x\cos^2 x + a^3 \cos^3 x)dx - \frac{4a}{\pi - 2}(x\sin x + \cos x)_0^{\pi/2} = 2|$

$\Rightarrow \frac{2}{3} + 3a.\frac{1}{3} + 3a^2.\frac{1}{3} + a^3.\frac{2}{3} - \frac{4a}{\pi - 2}\left\{\frac{\pi}{2} - 1\right\} = 2 \Rightarrow \frac{2a^2}{3} + a^2 + a + \frac{2}{3} - \frac{4a(\pi - 2)}{2(\pi - 2)} = 2$

$\Rightarrow \frac{2a^2}{3} + a^2 + a + \frac{2}{3} - 2a = 2 \Rightarrow 2a^3 + 3a^2 - 3a - 4 = 0 \Rightarrow (a + 1)(2a^2 + a - 4) = 0$

$\Rightarrow a = -1, \frac{-1 \pm \sqrt{33}}{4}$

2. Evaluate : सरल कीजिए :

$\int_0^{\pi} \sqrt{(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2}\, dx$

Ans. $\frac{\pi}{3} + 2\sqrt{3}$

Sol. $(\cos x + \cos 2x + \cos 3x)^2 + (\sin x + \sin 2x + \sin 3x)^2$

$= 3 + 2 (\cos x + \cos x + \cos 2x) = 3 + 4\cos x + 4\cos^2 x - 2 = 4\cos^2 x + 4\cos x + 1 = (2\cos x + 1)^2$

Hence I = $\int_0^{\pi} |2\cos x + 1|\, dx = \int_0^{2\pi/3} (1 + 2\cos x) - \int_{2\pi/3}^{\pi} (1 + 2\cos x)dx$

$= (x + 2\sin x)\Big|_0^{2\pi/3} - (x + 2\sin x)\Big|_{2\pi/3}^{\pi} = \frac{\pi}{3} + 2\sqrt{3}$

3. Let α & β be distinct positive roots of the equation tanx = 2x, then evaluate $\int_0^1 \sin(\alpha x).\sin(\beta x)dx$

माना α और β समीकरण tanx = 2x के भिन्न–भिन्न धनात्मक मूल है, तब सरल कीजिए $\int_0^1 \sin(\alpha x).\sin(\beta x)dx$

Ans. 0

Sol. $\tan\alpha = 2\alpha$ & $\tan\beta = 2\beta$

Now I = $\frac{1}{2}\int_0^1 \{\cos(\alpha - \beta)x - \cos(\alpha + \beta)x\}dx = \frac{1}{2}\left[\frac{\sin(\alpha - \beta)x}{\alpha - \beta} - \frac{\sin(\alpha + \beta)x}{\alpha + \beta}\right]_0^1$

I = $\frac{1}{2}\left\{\frac{\sin(\alpha - \beta)}{\alpha - \beta} - \frac{\sin(\alpha + \beta)}{\alpha + \beta}\right\}$

since $\tan\alpha = 2\alpha$ & $\tan\beta = 2\beta$

adding them we get $\frac{\sin(\alpha+\beta)}{\alpha+\beta} = 2\cos\alpha\,\cos\beta$

subtracting them we get $\frac{\sin(\alpha-\beta)}{\alpha-\beta} = 2\cos\alpha\,\cos\beta$

Hence $I = 0$

Hindi. $\tan\alpha = 2\alpha$ & $\tan\beta = 2\beta$

अब $I = \frac{1}{2}\int_0^1 \{\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\}dx = \frac{1}{2}\left[\frac{\sin(\alpha-\beta)x}{\alpha-\beta} - \frac{\sin(\alpha+\beta)x}{\alpha+\beta}\right]_0^1$

$I = \frac{1}{2}\left\{\frac{\sin(\alpha-\beta)}{\alpha-\beta} - \frac{\sin(\alpha+\beta)}{\alpha+\beta}\right\}$

चूँकि $\tan\alpha = 2\alpha$ तथा $\tan\beta = 2\beta$

उनको जोड़ने पर $\frac{\sin(\alpha+\beta)}{\alpha+\beta} = 2\cos\alpha\,\cos\beta$

उनको घटाने पर $\frac{\sin(\alpha-\beta)}{\alpha-\beta} = 2\cos\alpha\,\cos\beta$

अतः $I = 0$

4. Evaluate सरल कीजिए :

$$\lim_{a\to\left(\frac{\pi}{2}\right)^-} \int_0^a (\cos x)\ln(\cos x)dx$$

Ans. $\ell n2 - 1$

Sol. $\lim_{a\to\left(\frac{\pi}{2}\right)^-} \int_0^a (\cos x)\ln(\cos x)dx = \lim_{a\to\left(\frac{\pi}{2}\right)^-} \left\{\ell n(\cos x).\sin x\Big|_0^a + \int_0^a \frac{\sin x}{\cos x}.\sin x dx\right\}$

$= \lim_{a\to\left(\frac{\pi}{2}\right)^-} \{\sin a\ \ell n(\cos a) + \ell n(\sec a + \tan a) - \sin a\}$

$= \lim_{a\to\left(\frac{\pi}{2}\right)^-} \{\sin a.\ell n(\cos a) + \ell n(1 + \sin a) - \ell n(\cos a) - \sin a\}$

$= (\ell n2 - 1) + \lim_{t\to 0^+} (\sqrt{1-t^2} - 1).\ell n t$ where जहाँ $t = \cos a$

$= \ell n2 - 1 + \lim_{t\to 0^+} \frac{-t^2\ell n t}{\sqrt{1-t^2+1}} = \ell n2 - \lim_{t\to 0^+} \frac{\ell n t}{(1+t^2)} = \ell n2 - 1 + \frac{1}{2}\lim_{t\to 0^+} \frac{t^3.1}{2\cdot t} = \ell n2 - 1 + \frac{1}{4}\lim_{t\to 0^+} t^2 = \ell n2 - 1$

5. Find the value of $a(0 < a < 1)$ for which the following definite integral is minimized.

$$\int_0^\pi |\sin x - ax|\,dx$$

$a(0 < a < 1)$ का मान ज्ञात कीजिए जिसके लिए निश्चित समाकलन का मान न्यूनतम हो।

$$\int_0^{\pi} |\sin x - ax|\, dx$$

Ans. $a = \frac{\sqrt{2}}{\pi}\sin\left(\frac{\pi}{\sqrt{2}}\right)$

Sol. Let a ∈ (0, π), such that sinα = aα. माना a ∈ (0, π) इस प्रकार है कि sinα = aα.

$\therefore \quad \int_0^{\pi} |\sin x - ax|\, dx = \int_0^{\alpha} (\sin x - ax)\, dx + \int_{\alpha}^{\pi} (ax - \sin x)\, dx$

$\therefore \quad f(\alpha) = 1 - \cos\alpha - \frac{a\alpha^2}{2} + \frac{a(\pi^2 - \alpha^2)}{2} - 1 - \cos\alpha$

$f(\alpha) = \frac{a\alpha^2}{2} - a\alpha^2 - 2\cos\alpha$

$f(\alpha) = \frac{\pi^2 \sin\alpha}{2\alpha} - a\sin\alpha - 2\cos\alpha - 2\sin\alpha \left(\text{substituting } a = \frac{\sin\alpha}{\alpha}\right)\left(a = \frac{\sin\alpha}{\alpha} \text{ घटाने पर}\right)$

$f'(\alpha) = \frac{\pi^2}{2}\left\{\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2}\right\} - \alpha\cos\alpha - \sin\alpha + 2\sin\alpha$

$= \frac{\pi^2}{2}\frac{(\alpha\cos\alpha - \sin\alpha)}{\alpha^2} - (\alpha\cos\alpha - \sin\alpha) = (\alpha\cos\alpha - \sin\alpha)\left(\frac{\pi^2 - 2\alpha^2}{2\alpha^2}\right)$

Now (αcosα – sinα) is decreasing in (0, π) and never equals zero in (0, π)

Hence $f'(\alpha) = 0 \quad \Rightarrow \quad \alpha = \frac{\pi}{\sqrt{2}} \quad \Rightarrow \quad \alpha = \frac{\pi}{\sqrt{2}}$ as $\alpha > 0$

Morever f ' (α) changes its sign from –ve to +ve at $\frac{\pi}{\sqrt{2}}$, hence minima occurs at this value of α.

So $\sin\left(\frac{\pi}{\sqrt{2}}\right) = \frac{a\pi}{\sqrt{2}} \quad \Rightarrow \quad a = \frac{\sqrt{2}}{\pi}\sin\left(\frac{\pi}{\sqrt{2}}\right)$

Hindi माना a ∈ (0, π) इस प्रकार है कि sinα = aα.

$\therefore \quad \int_0^{\pi} |\sin x - ax|\, dx = \int_0^{\alpha} (\sin x - ax)\, dx + \int_{\alpha}^{\pi} (ax - \sin x)\, dx$

$\therefore \quad f(\alpha) = 1 - \cos\alpha - \frac{a\alpha^2}{2} + \frac{a(\pi^2 - \alpha^2)}{2} - 1 - \cos\alpha$

$f(\alpha) = \frac{a\alpha^2}{2} - a\alpha^2 - 2\cos\alpha$

$f(\alpha) = \frac{\pi^2 \sin\alpha}{2\alpha} - a\sin\alpha - 2\cos\alpha - 2\sin\alpha \left(\text{substituting } a = \frac{\sin\alpha}{\alpha}\right)\left(a = \frac{\sin\alpha}{\alpha} \text{ घटाने पर}\right)$

$f'(\alpha) = \frac{\pi^2}{2}\left\{\frac{\alpha\cos\alpha - \sin\alpha}{\alpha^2}\right\} - \alpha\cos\alpha - \sin\alpha + 2\sin\alpha$

$= \frac{\pi^2}{2}\frac{(\alpha\cos\alpha - \sin\alpha)}{\alpha^2} - (\alpha\cos\alpha - \sin\alpha) = (\alpha\cos\alpha - \sin\alpha)\left(\frac{\pi^2 - 2\alpha^2}{2\alpha^2}\right)$

अब (αcosα – sinα) (0, π) मे ह्रासमान है तथा (0, π) मे कभी भी शून्य नही है

अतः $f'(\alpha) = 0 \quad \Rightarrow \quad \alpha = \frac{\pi}{\sqrt{2}} \quad \Rightarrow \quad \alpha = \frac{\pi}{\sqrt{2}}$ as $\alpha > 0$

यहाँ तक कि f ' (α), इसका चिन्ह $\alpha = \frac{\pi}{\sqrt{2}}$ पर –ve से +ve बदलता है अतः α पर निम्निष्ठ रखेगा।

इसलिए $\sin\left(\frac{\pi}{\sqrt{2}}\right) = \frac{a\pi}{\sqrt{2}} \Rightarrow \quad a = \frac{\sqrt{2}}{\pi}\sin\left(\frac{\pi}{\sqrt{2}}\right)$

6. Find the $\lim\limits_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{\frac{1}{n}}$

where ${}^iC_j$ is a binomial coefficient which means $\frac{i.(i-1)....(i-j+1)}{j.(j-1)....2.1}$

$\lim\limits_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{\frac{1}{n}}$ ज्ञात कीजिए।

जहाँ ${}^iC_j$ द्विपद गुणांक है जिसका अर्थ $\frac{i.(i-1)....(i-j+1)}{j.(j-1)....2.1}$

Ans. $\frac{27}{16}$

Sol. $\lim\limits_{n\to\infty}\left(\frac{{}^{3n}C_n}{{}^{2n}C_n}\right)^{\frac{1}{n}} \quad \Rightarrow \lim\limits_{n\to\infty}\left\{\frac{(2n+1)(2n+2)........(3n)}{(n+1)(n+2)........2n}\right\}^{\frac{1}{n}}$

$= \lim\limits_{n\to\infty}\left\{\prod_{r=1}^{n}\frac{\left(2+\frac{r}{n}\right)}{\left(1+\frac{r}{n}\right)}\right\}^{\frac{1}{n}} = p \quad (\text{say})$ माना $= \ell np = \lim\limits_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\left\{\ell n\left(2+\frac{r}{n}\right)-\ell n\left(1+\frac{r}{n}\right)\right\}$

$= \int_0^1\{\ell n(2+r)-\ell n(1+x)\}dx$

$= x\ell n\left(\frac{2+x}{1+x}\right)\Big|_0^1 - \int_0^1 x\left(\frac{1}{x+2}-\frac{1}{x+1}\right)dx = x\ell n\left(\frac{3}{x}\right) + \left|\int_0^1 x\left(\frac{x+2-2}{x+2}-\frac{x+1-1}{x+1}\right)dx\right.$

$= \ell n\left(\frac{3}{2}\right) - \int_0^1\left\{\frac{x+2-2}{x+2}-\frac{x+1-1}{x+1}\right\}dx = \ell n\left(\frac{3}{2}\right) + 2\int_0^1\frac{dx}{x+2} - \int_0^1\frac{dx}{x+1} = \ell n\left(\frac{3}{2}\right) + 2\ell n\left(\frac{3}{2}\right) - \ell n2$

$= \ell n\left(\frac{27}{8}\right) - \ell n2 = \ell n\left(\frac{27}{16}\right)$

$p = \frac{27}{16}$

7. Show that $\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{\ell nx}{x}dx = \ell na.\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{dx}{x}$

दर्शाइए कि $\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{\ell nx}{x}dx = \ell na.\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right).\frac{dx}{x}$

Sol. $I = \int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right) dx$

Let $x = \frac{a}{t}$ ⇒ $dx = -\frac{a}{t^2}\,dt$ ⇒ $I = \int_0^\infty f\left(t+\frac{1}{t}\right)\frac{\ell n\left(\frac{a}{t}\right)}{a/t}\cdot\left(\frac{a}{t^2}\right) dt$

Again put $t = \frac{z}{a}$ ⇒ $dt = \frac{dz}{a}$ ⇒ $I = \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{\ell n\left(\frac{a^2}{z}\right)}{\frac{z}{a}}\,\frac{dz}{a}$

$= \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{[2\ell na - \ell nz]}{z}\,dz = \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{2\ell na}{z}\,dz - \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{\ell nz}{z}\,dz = 2\ \ell na \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{dz}{z} - I$

⇒ $2I = 2\ell na \int_0^\infty f\left(\frac{a}{z}+\frac{z}{a}\right)\cdot\frac{dz}{z}$ ⇒ $I = \ell na \int_0^\infty f\left(\frac{a}{z}+\frac{z}{a}\right)\cdot\frac{dz}{z} = \ell na \int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right)\cdot\frac{dx}{x}$

Hindi $I = \int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right) dx$

माना $x = \frac{a}{t}$ ⇒ $dx = -\frac{a}{t^2}\,dt$ ⇒ $I = \int_0^\infty f\left(t+\frac{1}{t}\right)\frac{\ell n\left(\frac{a}{t}\right)}{a/t}\cdot\left(\frac{a}{t^2}\right) dt$

पुनः $t = \frac{z}{a}$ रखने पर ⇒ $dt = \frac{dz}{a}$ ⇒

$I = \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{\ell n\left(\frac{a^2}{z}\right)}{\frac{z}{a}}\,\frac{dz}{a} = \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{[2\ell na - \ell nz]}{z}\,dz$

$= \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{2\ell na}{z}\,dz - \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{\ell nz}{z}\,dz = 2\ \ell na \int_0^\infty f\left(\frac{z}{a}+\frac{a}{z}\right)\frac{dz}{z} - I$

⇒ $2I = 2\ell na \int_0^\infty f\left(\frac{a}{z}+\frac{z}{a}\right)\cdot\frac{dz}{z}$ ⇒ $I = \ell na \int_0^\infty f\left(\frac{a}{z}+\frac{z}{a}\right)\cdot\frac{dz}{z} = \ell na \int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right)\cdot\frac{dx}{x}$

8. Evaluate सरल कीजिए $\lim\limits_{n\to\infty} n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} (2014\sin x + 2015\cos x)\,|x|\,dx$

Ans. 2015

Sol. $\lim\limits_{n\to\infty} n^2 \int_{-\frac{1}{n}}^{\frac{1}{n}} (2014\sin x + 2015\cos x)\,|x|\,dx = L$ (say) माना

Let माना $I = \int_{-\frac{1}{n}}^{\frac{1}{n}} (2014\sin x + 2015\cos x)\,|x|\,dx = 4030\int_0^{\frac{1}{n}} x\cos x\,dx = 4030(x\sin x + \cos x)\Big|_0^{1/n}$

$$= 4030\left\{\frac{1}{n}\sin\left(\frac{1}{n}\right)+\cos\left(\frac{1}{n}\right)-1\right\}$$

Now अब $L = \lim\limits_{n\to\infty} 4030\left\{n\sin\frac{1}{n}+n^2\left(\cos\left(\frac{1}{n}\right)-1\right)\right\} = 4030\ \left\{1-\frac{1}{2}\right\} = 2015$

9. Let sequence $\{a_n\}$ be defined as

$$a_1 = \frac{\pi}{4}\ ,\ a_n = \int_0^{\frac{1}{2}}(\cos(\pi x)+a_{n-1})\ \cos\pi x\ dx,\ (n = 2, 3, 4, ...)$$

then evaluate $\lim\limits_{n\to\infty} a_n$

माना अनुक्रम $\{a_n\}$ परिभाषित इस प्रकार है कि

$$a_1 = \frac{\pi}{4},\ a_n = \int_0^{\frac{1}{2}}(\cos(\pi x)+a_{n-1})\ \cos\pi x\ dx,\ (n = 2, 3, 4, ...)$$

तब सरल कीजिए $\lim\limits_{n\to\infty} a_n$

Ans. $\dfrac{\pi}{4(\pi-1)}$

Sol. $a_n = \displaystyle\int_0^{\frac{1}{2}}\cos^2(\pi x)\,dx + a_{n-1}\int_0^{\frac{1}{2}}\cos(\pi x)\,dx$

Let माना $\pi x = t$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\frac{\pi}{2}}\cos^2 t dt + \frac{a_{n-1}}{\pi}\int_0^{\frac{\pi}{2}}\cos t\,dt\right\} \Rightarrow a_n = \frac{1}{4}+\frac{a_{n-1}}{\pi} \Rightarrow \ell = \frac{1}{4} + \frac{\ell}{\pi}$$

$$\Rightarrow \lim_{n\to\infty} a_n = \ell = \frac{\pi}{4(\pi-1)}$$

10. Find f(x) if it satisfies the relation $f(x) = e^x + \displaystyle\int_0^1 (x+ye^x)\ f(y)\ dy.$

f(x) ज्ञात कीजिए यदि यह संबंध $f(x) = e^x + \displaystyle\int_0^1 (x+ye^x)\ f(y)\ dy$ को संतुष्ट करता है।

Ans. $\dfrac{-3e^x}{2(e-1)} - 3x$

Sol. $f(x) = e^x + x\displaystyle\int_0^1 f(y)\ \ dy + e^x\int_0^1 y\ \ \ f(y)\ \ dy$

where (जहाँ) $A = \displaystyle\int_0^1 f(y)\ \ dy$ & तथा $B = \displaystyle\int_0^1 y\ \ \ f(y)\ \ dy$

Now (अब), $A = \displaystyle\int_0^1 (e^y + Ay + Be^y)\ \ dy\ = \left[e^y + A.\frac{y^2}{2} + Be^y\right]_0^1 = e + \frac{A}{2} + B = e - 1 + A/2 + Bc - B$

$$B = \int_0^1 y\ \ (e^y + Ay + Be^y)\ \ dy\ = \int_0^1 \left(ye^y(1+B)\ \ +Ay^2\right)\ \ dy$$

$$B = (1 + B)\ (ye^y - e^y)_0^1 + A\left(\frac{y^3}{3}\right)_0^1 = -\ (1+B) + \frac{A}{3}$$

From (i) & (ii) (i) तथा (ii) से

11. Evaluate : $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1+x^2}\right) dx.$

मान ज्ञात कीजिए : $\int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{2x}{1+x^2}\right)$ **Ans.** $\frac{\pi}{4}\ \ell n\left(2+\sqrt{3}\right) + \frac{\pi^2}{12} - \frac{\pi}{\sqrt{3}}$

Sol. $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{-2x}{1+x^2}\right) dx$ Applying $\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$

$$I = \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx - I$$

i.e. $2I = \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\ dx = 2\pi \int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}\ dx \Rightarrow$ $I = \pi \int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx$

$$= \frac{\pi}{2} \int_0^{1/\sqrt{3}} \left(-2 + \frac{1}{1-x^2} + \frac{1}{1+x^2}\right)\ dx$$

$$= \frac{\pi}{2} \int_0^{1/\sqrt{3}} \left(-2 + \frac{1}{1-x^2} + \frac{1}{1+x^2}\right)\ dx\ = \frac{\pi}{2}\left[-2x + \frac{1}{2.1}\ \ell n\ \frac{1+x}{1-x} + \tan^{-1} x\right]_0^{1/\sqrt{3}}$$

$$= \frac{\pi}{2}\left[-\frac{2}{\sqrt{3}} + \frac{1}{2}\ \ell n\ \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\pi}{6}\right] = \frac{\pi}{4}\ \ell n\left(2+\sqrt{3}\right) + \frac{\pi^2}{12} - \frac{\pi}{\sqrt{3}}$$

Hindi $I = \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} \cos^{-1}\left(\frac{-2x}{1+x^2}\right) dx$ $\int_a^b f(x)\ dx = \int_a^b f(a+b-x)\ dx$ का प्रयोग करने पर

$$I = \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx - I$$

अर्थात् $2I = \pi \int_{-1/\sqrt{3}}^{1/\sqrt{3}} \frac{x^4}{1-x^4}\ dx = 2\pi \int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4}\ dx \Rightarrow$ $I = \pi \int_0^{1/\sqrt{3}} \frac{x^4}{1-x^4} dx$

$$= \frac{\pi}{2} \int_0^{1/\sqrt{3}} \left(-2 + \frac{1}{1-x^2} + \frac{1}{1+x^2}\right)\ dx$$

$$= \frac{\pi}{2} \int_0^{1/\sqrt{3}} \left(-2 + \frac{1}{1-x^2} + \frac{1}{1+x^2}\right)\ dx\ = \frac{\pi}{2}\left[-2x + \frac{1}{2.1}\ \ell n\ \frac{1+x}{1-x} + \tan^{-1} x\right]_0^{1/\sqrt{3}}$$

$$= \frac{\pi}{2}\left[-\frac{2}{\sqrt{3}} + \frac{1}{2}\ \ell n\ \frac{\sqrt{3}+1}{\sqrt{3}-1} + \frac{\pi}{6}\right] = \frac{\pi}{4}\ \ell n\left(2+\sqrt{3}\right) + \frac{\pi^2}{12} - \frac{\pi}{\sqrt{3}}$$

12. Evaluate $\int_0^1 \frac{1}{\left(5+2x-2x^2\right)\ \left(1+e^{(2-4x)}\right)} dx$

मान ज्ञात कीजिए $\int_0^1 \frac{1}{\left(5+2x-2x^2\right)\ \left(1+e^{(2-4x)}\right)} dx$ **Ans.** $\frac{1}{2}\ \frac{1}{\sqrt{11}}\ \ell n\ \frac{\sqrt{11}+1}{\sqrt{11}-1}$

Sol. $\text{I} = \int_0^1 \frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ ...(1)

$$= \int_0^1 \frac{dx}{[5+2\ \ (1-x)-2\ \ (1-x)^2]\ \ [1+e^{2\ -\ 4(1-x)}]}$$ By $\int_a^b f(x)\ \ dx\ = \int_a^b f(a+b-x)\ \ dx$ (से)

$$= \int_0^1 \frac{dx}{(5+2x-2x^2)\ \ (1+e^{-2+\ 4x})} = \int_0^1 \frac{e^{2-4x}\ \ \ \ dx}{(5+2x-2x^2)\ \ (1+e^{2-\ 4x})}$$ ...(2)

(1) + (2)

$$\Rightarrow 2\text{I} = \int_0^1 \frac{e^{2-4x}+1\ \ \ \ dx}{(5+2x-2x^2)\ \ (1+e^{2-\ 4x})} = \int_0^1 \frac{dx}{(5+2x-2x^2)}$$

$$= -\frac{1}{2}\int_0^1 \frac{dx}{\left(x^2-x-\frac{5}{2}\right)} = -\frac{1}{2}\int_0^1 \frac{dx}{\left(x-\frac{1}{2}\right)^2-\left(\frac{\sqrt{11}}{2}\right)^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{11}}\left[\ell n\left(\frac{\frac{\sqrt{11}}{2}+x-\frac{1}{2}}{\frac{\sqrt{11}}{2}-x+\frac{1}{2}}\right)\right]_0^1 = \frac{1}{2}\,\frac{1}{\sqrt{11}}\ \ell n\left(\frac{\sqrt{11}+1}{\sqrt{11}-1}\right)$$

13. Prove that for any positive integer k ;

$\frac{\sin 2k\,x}{\sin x}$ = 2 [cos x + cos 3x +..... + cos (2k – 1)x]. Hence prove that;

$$\int_0^{\pi/2} \sin\,(2kx).\ \cot x\ dx = \frac{\pi}{2}.$$

किसी धनात्मक पूर्णांक k के लिए सिद्ध करो कि –

$\frac{\sin 2k\,x}{\sin x}$ = 2 [cos x + cos 3x +..... + cos (2k – 1)x]. इस प्रकार सिद्ध कीजिये कि –

$$\int_0^{\pi/2} \sin 2k\ x.\ \cot x\ dx = \frac{\pi}{2}.$$

Sol. 2 sin x [cosx + cos 3x + ... + cos (2k – 1)]

= 2sin x cos x + 2 sin x cos 3x + ... + 2 sin x cos (2k – 1)x

= sin 2x + (sin 4x – sin 2x) + (sin 6x – sin 4x) + ... + {sin 2kx – sin (2kx – 2x)}

= sin 2kx

∴ 2[cos x + cos 3x + ... + cos(2k – 1) x] = $\frac{\sin 2kx}{\sin x}$ ......(1)

Now, अब sin 2kx cot x = $\frac{\sin 2k\ \ x}{\sin x}$. cos x

= 2 cos x[cosx + cos 3x + cos 5x + ... + cos (2k – 1) x] [using (1)] [(1) का प्रयोग करने पर]

= 2 cos²x + 2 cos 3x cosx + 2 cos 5x cos x + ... + 2 cos (2k – 1) x cos x

= (1 + cos 2x) + (cos 4x + cos 2x) + (cos 6x + cos 4x) + ... + (cos 2kx + cos(2k – 2) x)

= 1 + 2 {cos 2x + cos 4x + cos 6x + cos(2k – 2) x} + cos 2k x

$$\therefore \int_0^{\pi/2} \sin 2kx \cot x\, dx = \int_0^{\pi/2} 1\ \ .\ \ dx + 2\int_0^{\pi/2} [\cos 2x + ... + \cos(2k-2)x]dx + \int_0^{\pi/2} \cos 2kx dx = \frac{\pi}{2}$$

14. Prove that $\text{Limit}_{n\to\infty} \frac{1}{n}\left[\cos^{2p}\frac{\pi}{2n} + \cos^{2p}\frac{2\pi}{2n} + \cos^{2p}\frac{3\pi}{2n} + ... + \cos^{2p}\frac{\pi}{2}\right] = \prod_{r=1}^{p}\frac{2\,r-1}{2\ \ r}$,

where Π denotes the continued product and p ∈ N.

सिद्ध कीजिए कि $\mathop{\text{Limit}}_{n\to\infty} \frac{1}{n}\left[\cos^{2p}\frac{\pi}{2n} + \cos^{2p}\frac{2\pi}{2n} + \cos^{2p}\frac{3\pi}{2n} + ... + \cos^{2p}\frac{\pi}{2}\right] = \prod_{r=1}^{p}\frac{2r-1}{2r}$,

जहाँ Π क्रमागत गुणन को प्रदर्शित करता है तथा $p \in N$

Sol. $\mathop{\text{Limit}}_{n\to\infty} \frac{1}{n}\left[\cos^{2p}\frac{\pi}{2n} + \cos^{2p}\frac{2\pi}{2n} + \cos^{2p}\frac{3\pi}{2n} + ... + \cos^{2p}\frac{n\pi}{2n}\right]$

$= \int_0^1 \cos^{2p}\frac{\pi x}{2}dx$, $\frac{\pi x}{2} = t$

$= \int_0^{\pi/2} \cos^{2p} t \;.\; dt = \frac{2}{\pi}\int_0^{\pi/2} \cos^{2p} t \;\; dt$ now by reduction formula (समानयन सूत्र द्वारा)

$= \frac{2}{\pi}\left[\frac{2p-1}{2p} \cdot \frac{2p-3}{2p-2} \cdot .... \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right] = \prod_{r=1}^{p}\frac{2r-1}{2r}$

15. If n > 1, evaluate $\int_0^\infty \frac{dx}{\left(x+\sqrt{1+x^2}\right)^n}$

यदि $n > 1$ हो, तो $\int_0^\infty \frac{dx}{\left(x+\sqrt{1+x^2}\right)^n}$ का मान ज्ञात कीजिए। **Ans.** $\frac{n}{n^2-1}$

Sol. $I = \int_0^\infty \frac{dx}{\left(x+\sqrt{1+x^2}\right)^n}$ Put $x + \sqrt{(1+x^2)} = t$...(1)

$\therefore \left(1+\frac{x}{\sqrt{1+x^2}}\right) dx = dt$

when $x \to 0$ then $t \to 1$

$x \to \infty$ then $t \to \infty$

$\Rightarrow \frac{tdx}{\sqrt{1+x^2}} = dt \quad \Rightarrow \quad dx = \frac{\sqrt{(1+x^2)}}{t}\,dt$

and $\sqrt{1+x^2} - x = \frac{1}{t}$ **...(2)**

Adding (1) and (2) we get

$2\sqrt{1+x^2} = t + \frac{1}{t} \quad \Rightarrow \quad \sqrt{1+x^2} = \frac{t^2+1}{2t} \quad \therefore \quad dx = \frac{(t^2+1)}{2t^2}\,dt$

Hence $I = \int_1^\infty \frac{(t^2+1)\;\; dt}{2t^2t^n} = \frac{1}{2}\int_1^\infty \left(\frac{1}{t^n}+\frac{1}{t^{n+2}}\right) dt$

$= \frac{1}{2}\left[-\frac{1}{(n-1)\;\; t^{n-1}} - \frac{1}{(n+1)\;\; t^{n+1}}\right]_1^\infty = \frac{1}{2}\left[-(0-0)-\left(-\frac{1}{n-1}-\frac{1}{n+1}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{n-1}+\frac{1}{n+1}\right]$ $(\because n > 1)$

$= \frac{n}{n^2-1}$

Hindi. $I = \int_0^\infty \frac{dx}{\left(x+\sqrt{1+x^2}\right)^n}$

$x + \sqrt{(1+x^2)} = t$ रखने पर ...(1)

$\therefore \left(1 + \frac{x}{\sqrt{1+x^2}}\right) dx = dt$

जब $x \to 0$, तब $t \to 1$ $x \to \infty$, तब $t \to \infty$

$\Rightarrow \frac{t dx}{\sqrt{1+x^2}} = dt$ $\Rightarrow dx \frac{\sqrt{(1+x^2)}}{t} = dt$ तथा $\sqrt{1+x^2} - x = \frac{1}{t}$ .....(2)

(1) तथा (2) को जोड़ने पर $2\sqrt{1+x^2} = t + \frac{1}{t}$ $\Rightarrow$ $\sqrt{1+x^2} = \frac{t^2+1}{2t}$

$\therefore \quad dx = \frac{(t^2+1)}{2t} dt$

अत : $I = \int_0^\infty \frac{(t^2+1)\ dt}{2t^2 t^n} = \frac{1}{2}\int_0^\infty \left(\frac{1}{t^n} + \frac{1}{t^{n+2}}\right) dt$

$= \frac{1}{2}\left[-\frac{1}{(n-1)\ t^{n-1}} - \frac{1}{(n+1)\ t^{n+1}}\right]_1^\infty = \frac{1}{2}\left[-(0-0) - \left(-\frac{1}{n-1} - \frac{1}{n+1}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{n-1} + \frac{1}{n+1}\right]$ $(\because n > 1)$ $= \frac{n}{n^2-1}$

16. Let f(x) be a continuous function ∀ x ∈ R, except at x = 0 such that $\int_0^a f(x)dx$, a ∈ R⁺ exists. If $g(x) = \int_x^a \frac{f(t)}{t}$ dt, prove that $\int_0^a g(x)\ dx = \int_0^a f(x)\ dx$

माना f(x), x = 0 के अलावा x के प्रत्येक वास्तविक मान के लिए एक सतत् फलन इस प्रकार है कि $\int_0^a f(x)dx$, a ∈ R⁺ अस्तित्व में है। यदि $g(x) = \int_x^a \frac{f(t)}{t}$ dt हो, तो सिद्ध करो कि $\int_0^a g(x)\ dx = \int_0^a f(x)\ dx$

Sol.

$g(x) = \int_x^a \frac{f(t)}{t} dt = -\int_a^x \frac{f(t)}{t} dt.$

$\therefore \quad g'(x) = -\frac{f(x)}{x}$ $\Rightarrow$ $f(x) = -x\, g'(x)$

$\Rightarrow \int_0^a f(x)\ dx = -\int_0^a x\ g'(x)\ dx$ $= -[x\ g\ (x)]_0^a + \int_0^a g\ (x)\ dx$

$=$ $-a\, g(a) + \int_0^a g\ (x)\, dx$ $= \int_0^a g\ (x)\ dx$ $\left[\because g\ (a) = \int_a^a \frac{f(t)}{t}\ dt = 0\right]$

17. Given that $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{\log_e(n^2+r^2) - 2\log_e n}{n} = \log_e 2 + \frac{\pi}{2} - 2$, then

evaluate : $\lim_{n\to\infty} \frac{1}{n^{2m}} [(n^2 + 1^2)^m (n^2 + 2^2)^m \ldots\ldots (2n^2)^m]^{1/n}$.

दिया हुआ है कि $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{\log_e(n^2+r^2) - 2\log_e n}{n} = \log_e 2 + \frac{\pi}{2} - 2$, तो

$\lim_{n\to\infty} \frac{1}{n^{2m}} [(n^2 + 1^2)^m (n^2 + 2^2)^m \ldots\ldots (2n^2)^m]^{1/n}$. का मान ज्ञात कीजिए।

Ans. $\left(\frac{2\sqrt{e^\pi}}{e^2}\right)^m$

Sol. Let $L = \lim_{n\to\infty} \frac{1}{n^{2m}} [(n^2 + 1^2)^m (n^2 + 2^2)^m .... (2n^2)^m]^{1/n}$.

$$\Rightarrow \quad L = \lim_{n\to\infty} \left(\frac{(n^2+1^2)(n^2+2^2)....(2n^2)}{n^{2n}}\right)^{\frac{m}{n}} \Rightarrow \lim_{n\to\infty} \log_e L = \log_e \left[\frac{1}{n^{2n}}\{(n^2+1^2)(n^2+2^2)....(2n^2)\}\right]^{\frac{m}{n}}$$

$$= \lim_{n\to\infty} \frac{m}{n}\left(\left(\sum_{k=1}^{n}\log_e(n^2+k^2)\right) - 2n\log_e n\right) = \lim_{n\to\infty} \frac{m}{n}\sum_{k=1}^{n}(\log_e(n^2+k^2) - 2\log_e n)$$

$$= \lim_{n\to\infty} m\sum_{k=1}^{n}\frac{\log_e(n^2+k^2) - 2\log_e n}{n}$$

$$\Rightarrow \quad \log_e L = m\left(\ell n2+\frac{\pi}{2}-2\right) \quad \Rightarrow \quad L = e^{m\left(\ell n2+\frac{\pi}{2}-2\right)} = \left(\frac{2\sqrt{e^{\pi}}}{e^2}\right)^m$$

Hindi माना $L = \lim_{n\to\infty} \frac{1}{n^{2m}} [(n^2 + 1^2)^m (n^2 + 2^2)^m .... (2n^2)^m]^{1/n}$.

$$\Rightarrow \quad L = \lim_{n\to\infty} \left(\frac{(n^2+1^2)(n^2+2^2)....(2n^2)}{n^{2n}}\right)^{\frac{m}{n}} \Rightarrow \lim_{n\to\infty} \log_e L = \log_e \left[\frac{1}{n^{2n}}\{(n^2+1^2)(n^2+2^2)....(2n^2)\}\right]^{\frac{m}{n}}$$

$$= \lim_{n\to\infty} \frac{m}{n}\left(\left(\sum_{k=1}^{n}\log_e(n^2+k^2)\right) - 2n\log_e n\right) = \lim_{n\to\infty} \frac{m}{n}\sum_{k=1}^{n}(\log_e(n^2+k^2) - 2\log_e n)$$

$$= \lim_{n\to\infty} m\sum_{k=1}^{n}\frac{\log_e(n^2+k^2) - 2\log_e n}{n}$$

$$\Rightarrow \quad \log_e L = m\left(\ell n2+\frac{\pi}{2}-2\right) \quad \Rightarrow \quad L = e^{m\left(\ell n2+\frac{\pi}{2}-2\right)} = \left(\frac{2\sqrt{e^{\pi}}}{e^2}\right)^m$$

18. For a natural number n, let $a_n = \int_0^{\pi/4} (\tan x)^{2n} dx$

Now answer the following questions :

(1) Express a_{n+1} in terms of a_n

(2) Find $\lim_{n\to\infty} a_n$

(3) Find $\lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k-1} (a_k + a_{k-1})$

एक प्राकृत संख्या n के लिए, माना $a_n = \int_0^{\pi/4} (\tan x)^{2n} dx$

अब निम्नलिखित प्रश्नों के उत्तर दीजिए

(1) a_{n+1} को a_n के पदों मे व्यक्त कीजिए।

(2) $\lim_{n\to\infty} a_n$ ज्ञात कीजिए।

(3) $\lim_{n\to\infty} \sum_{k=1}^{n} (-1)^{k-1} (a_k + a_{k-1})$ ज्ञात कीजिए ।

Ans. (1) $\frac{1}{2n+1} - a_n$ **(2) 0** **(3)** $\frac{\pi}{4}$

Sol. $a_1 = \int_0^{\pi/4} \tan^2 x dx = \int_0^{\pi/4} (\sec^2 x - 1)dx = (\tan x - x)_0^{\pi/4} = 1 - \frac{\pi}{4}$ and $a_0 = \frac{\pi}{4}$

$a_1 = \int_0^{\pi/4} \tan^2 x dx = \int_0^{\pi/4} (\sec^2 x - 1) \ dx = (\tan x - x)_0^{\pi/4} = 1 - \frac{\pi}{4}$ तथा $a_0 = \frac{\pi}{4}$

(1) $a_{n+1} = \int_0^{\pi/4} (\tan x)^{2n+2} dx = \int_0^{\pi/4} (\tan x)^{2n}(\sec^2 x - 1)\ dx = \int_0^{\pi/4} (\tan x)^{2n} \sec^2 x dx - a_n = \frac{1}{2n+1} - a_n$

(2) $\because a_{n+1} = \frac{1}{2n+1} - a_n$

Let (माना) $\lim\limits_{n\to\infty} a_n = a \Rightarrow \lim\limits_{n\to\infty} a_{n+1} = a$

$\Rightarrow \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \left(\frac{1}{2n+1} - a_n\right) \Rightarrow \quad a = 0 - a$

$\Rightarrow 2a = 0 \Rightarrow \quad a = 0$

(3) $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k-1} (a_k + a_{k-1})$

$= \lim\limits_{n\to\infty} \sum_{k=1}^{n} (-1)^{k+1}(a_k + a_{k-1}) = \lim\limits_{n\to\infty} [1 - (a_2 + a_1) + (a_3 + a_2) - (a_4 + a_3) + ... + (-1)^{n+1} (a_n + a_{n-1})]$

$= \lim\limits_{n\to\infty} [1 - a_1 + (-1)^{n+1} a_n] = 1 - a_1 + 0 = 1 - \left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4}$.

19. Given that $\lim\limits_{x\to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+t}} dt}{bx - \sin x} = 1$, then find the values of a and b

दिया हुआ है कि $\lim\limits_{x\to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+t}} dt}{bx - \sin x} = 1$, तो a तथा b के मान ज्ञात कीजिए।

Ans. a = 4, b = 1

Sol. $\lim\limits_{x\to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+t}} dt}{bx - \sin x} = 1$

since limit is of $\left(\frac{0}{0}\right)$ form

$\lim\limits_{x\to 0} \frac{\frac{x^2}{\sqrt{a+x}}}{b - \cos\ x} = 1$; $\lim\limits_{x\to 0} \frac{x^2}{\sqrt{a+x}\,(b - \cos x)} = 1$

As x → 0, Numerator → 0 but Denominator → $\sqrt{a}$ (b – 1)

But limit exists and is equal to 1

therefore b – 1 = 0 ⇒ b = 1

a0, otherwise limit will not exist.

$\therefore \lim\limits_{x\to 0} \frac{x^2}{\sqrt{a+x}\,(1-\cos x)} = 1 \Rightarrow \lim\limits_{x\to 0} \frac{1}{\sqrt{a+x}\left(\frac{1-\cos x}{x^2}\right)} = 1 \Rightarrow \frac{2}{\sqrt{a}} = 1 \Rightarrow a = 4$

Hindi. $\lim\limits_{x\to 0} \frac{\int_0^x \frac{t^2}{\sqrt{a+t}} dt}{bx - \sin x} = 1$

चूंकि सीमा $\left(\frac{0}{0}\right)$ रूप की है

$\lim\limits_{x\to0}\dfrac{\dfrac{x^2}{\sqrt{a+x}}}{b-\cos\ \ x}=1$; $\lim\limits_{x\to0}\dfrac{x^2}{\sqrt{a+x}\,(b-\cos x)}=1$

जैसे $x \to 0$, अंश $\to 0$ किन्तु हर $\to \sqrt{a}\ \ (b-1)$

किन्तु सीमा विद्यमान है तथा 1 के बराबर है

इसलिए $b-1=0 \Rightarrow \quad b=1$

a0, अन्यथा सीमा विद्यमान नहीं होगी।

$\therefore\ \lim\limits_{x\to0}\dfrac{x^2}{\sqrt{a+x}\,(1-\cos x)}=1 \quad\Rightarrow\quad \lim\limits_{x\to0}\dfrac{1}{\sqrt{a+x}\ \left(\dfrac{1-\cos x}{x^2}\right)}=1 \quad\Rightarrow\quad \dfrac{2}{\sqrt{a}}=1 \quad\Rightarrow\quad a=4$

20. Prove that m sin x + $\int\limits_0^x \sec^m t\ \ dt > (m+1)x\ \ \forall\ x \in \left(0,\ \dfrac{\pi}{2}\right)\ m \in N$

सिद्ध कीजिए कि m sin x + $\int\limits_0^x \sec^m t\ \ dt > (m+1)x\ \ \forall\ x \in \left(0,\ \dfrac{\pi}{2}\right)\ m \in N$

Sol. Let f(x) = m sin x + $\int\limits_0^x \sec^m t\ \ dt\ -(m+1)\ x$

f '(x) = m cos x + $\sec^m$ x – (m + 1)

Now apply A. M > G.M for

cosx, cosx,........., cosx, $\sec^m$ x, we get

cosx + cos x + ... + cos x + $\sec^m$ x > (m + 1)

So f(x) is increasing in $\left[0,\ \dfrac{\pi}{2}\right)$ $\Rightarrow$ f(x) > f(0) $\forall\ x \in \left(0,\ \dfrac{\pi}{2}\right)$

Hindi माना f(x) = m sin x + $\int\limits_0^x \sec^m t\ \ dt - (m+1)\ x$

f '(x) = m cos x + $\sec^m$ x – (m + 1)

A. M > G.M का प्रयोग करने पर

cosx, cosx,........., cosx, $\sec^m$ x

cosx + cos x + ... + cos x + $\sec^m$ x > (m + 1)

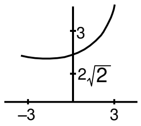
इसलिए f(x) वर्द्धमान है $\left[0,\ \dfrac{\pi}{2}\right)$ में $\Rightarrow$ f(x) > f(0) $\forall\ x \in \left(0,\ \dfrac{\pi}{2}\right)$

21. f(x) is differentiable function: g(x) is double differentiable function such that |f(x)| ≤ 1 and g(x) = f'(x). If $f^2(0)+g^2(0)=9$ then show that there exists some C ∈ (–3, 3) such that g(c) g" (c) < 0

यदि f(x) अवकलनीय फलन है। g(x) दो बार अवकलनीय फलन इस प्रकार है कि |f(x)| ≤ 1 तथा g(x) = f'(x). यदि $f^2(0)$ + $g^2(0)$ = 9 तब दर्शाईये कि तब कम से कम एक C ∈ (–3, 3) इस प्रकार विद्यमान है कि g(c) g" (c) < 0

Sol. $g(x) > 0\ \ ;\ \ g''(x) \ge 0\ \ ;\ \ \int\limits_{-3}^{3} g(x)\ \ dx = \int\limits_{-3}^{3} f'(x)\ \ dx\ = f(3)-f(-3)\ \ ;\ \ \int\limits_{-3}^{3} g(x)\ \ dx < 2$

From fig चित्र से .



$\int\limits_{-3}^{3} g(x)\ \ dx > 6\sqrt{2}$

so its not possible. Hence g"(x) < 0 for some x

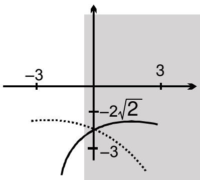
इसलिए यह संभव नहीं है अतः किसी x के लिए g"(x) < 0

H.P.

if यदि $g(x) > 0$; $g''(x) < 0$; No need कोई आवश्यकता नहीं

if यदि $g(x) < 0$; $g''(x) > 0$; No need कोई आवश्यकता नहीं

If यदि $g(x) < 0$; $g''(x) < 0$; $\int_{-3}^{3} g(x)\ dx \le f(3) - f(-3)$



$-2 < \int_{-3}^{3} g(x)\ dx < \sqrt{2}$; Again contradiation पुनः विरोधाभास

H.P.

22. Draw a graph of the function $f(x) = \cos^{-1}(4x^3 - 3x)$, $x \in [-1, 1]$ and find the area enclosed between the graph of the function and the x–axis as x varies from 0 to 1.
फलन $f(x) = \cos^{-1}(4x^3 - 3x)$, $x \in [-1, 1]$ का आरेख बनाइए तथा फलन के आरेख एवं x–अक्ष के मध्य x = 0 से 1 तक परिबद्ध क्षेत्र का क्षेत्रफल ज्ञात कीजिए।

Ans. $3\ (\sqrt{3} - 1)$ sq. units वर्ग इकाई

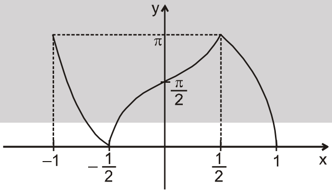
Sol. $f(x) = \cos^{-1}(4x^3 - 3x)$

$x = \cos\theta \quad \Rightarrow \quad \theta = \cos^{-1} x$

$f(x) = \cos^{-1}(\cos 3\theta)$

$\because \quad \theta \in [0, \pi] \quad \Rightarrow \quad 3\theta \in [0, 3\pi]$

$$f(x) = \begin{cases} 3\theta & , \quad 0 \le 3\theta \le \pi \\ 2\pi - 3\theta & , \quad \pi \le 3\theta \le 2\pi \\ 3\theta - 2\pi & , \quad 2\pi \le 3\theta \le 3\pi \end{cases}$$



Graph of y = f(x)

or या
$$f(x) = \begin{cases} 3\cos^{-1} x & , \quad \frac{1}{2} \le x \le 1 \\ 2\pi - 3\cos^{-1} x & , \quad -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1} x - 2\pi & , \quad -1 \le x \le -\frac{1}{2} \end{cases}$$

Area क्षेत्रफल $= \int_{0}^{1/2} (2\pi - 3\cos^{-1} x)\ dx + \int_{1/2}^{1} (3\cos^{-1} x)\ dx$

$= (2\pi x - 3x \cos^{-1} x + 3\,(1 - x^2)^{1/2}) \Big|_{0}^{1/2} + 3(x \cos^{-1} x - (1 - x^2)^{1/2}) \Big|_{1/2}^{1}$

$= 3\ (\sqrt{3} - 1)$

23. Consider a square with vertices at (1, 1), (– 1, 1), (–1, –1) and (1, –1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area.
शीर्षों (1, 1), (– 1, 1), (–1, –1) तथा (1, –1) वाला एक वर्ग है। वर्ग के अंदर के ऐसे सभी बिन्दुओं,

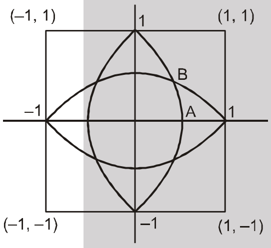
जो कि किसी भुजा के बजाय मूल बिन्दु के ज्यादा पास हो, वाला क्षेत्र S है। क्षेत्र S का रेखा चित्र बनाइए तथा इसका क्षेत्रफल ज्ञात कीजिए।

Ans. $\frac{1}{3}\left(16\sqrt{2}-20\right)$

Sol. S is region common to interior of

$y^2 = 1 - 2x$, $y^2 = 1 + 2x$, $x^2 = 1 - 2y$ and $x^2 = 1 + 2y$.

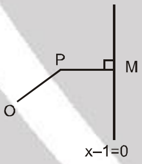
Here $A \equiv (1/2, 0)$, $B \equiv (\sqrt{2}-1,\ \sqrt{2}-1)$



Figure

Required area अभीष्ट क्षेत्रफल $= 4\int_{0}^{\sqrt{2}-1}\left(\frac{1-x^2}{2}\right)dx + 4\int_{\sqrt{2}-1}^{1/2}\sqrt{1-2x}\ dx$

$$= 2\left(\left(\sqrt{2}-1\right)-\frac{\left(\sqrt{2}-1\right)^3}{3}\right) - \frac{4}{3}\left(0 - \left(3-2\sqrt{2}\right)^{3/2}\right)$$



चित्र

$$= \frac{1}{3}\left(16\sqrt{2}-20\right)$$

24. If [x] denotes the greatest integer function. Draw a rough sketch of the portions of the curves $x^2 = 4\left[\sqrt{x}\right]$ y and $y^2 = 4\left[\sqrt{y}\right]$ x that lie within the square {(x, y) | 1 $\le$ x $\le$ 4, 1 $\le$ y < 4} Find the area of the part of the square that is enclosessd by the two curves and the line x + y = 3

यदि [x] महत्तम पूर्णांक फलन को प्रदर्शित करे तो वक्रों $x^2 = 4\left[\sqrt{x}\right]$ y तथा $y^2 = 4\left[\sqrt{y}\right]$ x का कच्चा चित्र बनाइऐ जो वर्ग {x, y) | 1 $\le$ x $\le$ 4, 1 $\le$ y < 4} के अन्दर स्थित है। वर्ग के उस भाग का क्षेत्रफल ज्ञात कीजिए जो दोनों वक्र एवं सरल रेखा x + y = 3 से परिबद्ध है।

Ans. $\frac{19}{6}$

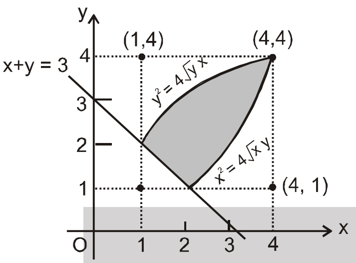
Sol. We have, $1 \le x \le 4$ and $1 \le y \le 4$

$\Rightarrow$ $1 \le \sqrt{X} \le 2$ and $1 \le \sqrt{y} \le 2$

$\left[\sqrt{x}\right] = 1$ and $\left[\sqrt{y}\right] = 1$ for all (x, y) lying with in the square.

Thus, $x^2 = 4y\left[\sqrt{x}\right]$ and $y^2 = 4\left[\sqrt{y}\right]x$

$\Rightarrow$ $x^2 = 4y$ and $y^2 = 4x$ when $1 < x, y < 4$ which could be plotted as :



Figure

Thus the required area :

$$= \int_1^2 (2\sqrt{x}-3+x)\ \ dx \ + \ \int_2^4 \left(2\sqrt{x}-\frac{x^2}{4}\right)\ dx \ = \left(\frac{4}{3}x^{3/2}-3x+\frac{x^2}{2}\right)_1^2 + \left(\frac{4}{3}x^{3/2}-\frac{x^3}{12}\right)_2^4$$

$$= \left(\frac{8\sqrt{2}}{3}-6+2\right) - \left(\frac{4}{3}-3+\frac{1}{2}\right) + \left(\frac{32}{3}-\frac{64}{12}\right) - \left(\frac{8\sqrt{2}}{3}-\frac{8}{12}\right) = \frac{19}{6} \text{ sq. units}$$

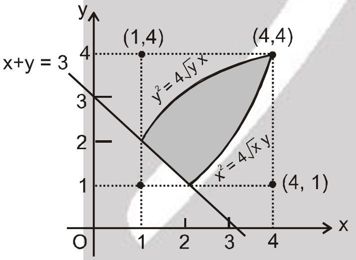
Hindi दिया है, $1 \le x \le 4$ तथा $1 \le y \le 4$

$\Rightarrow$ $1 \le \sqrt{X} \le 2$ तथा $1 \le \sqrt{y} \le 2$

$\left[\sqrt{x}\right] = 1$ तथा $\left[\sqrt{y}\right] = 1$ सभी (x, y) के लिए जो वर्ग के अन्दर स्थित है।

इसलिए $x^2 = 4y\left[\sqrt{x}\right]$ तथा $y^2 = 4\left[\sqrt{y}\right]x$

$\Rightarrow$ $x^2 = 4y$ तथा $y^2 = 4x$ जब $1 < x, y < 4$ जिसको निम्न प्रकार से आरेखित कर सकते है



Figure

अतः अभीष्ट क्षेत्रफल

$$= \int_1^2 (2\sqrt{x}-3+x)\ \ dx \ + \ \int_2^4 \left(2\sqrt{x}-\frac{x^2}{4}\right)\ dx \ = \left(\frac{4}{3}x^{3/2}-3x+\frac{x^2}{2}\right)_1^2 + \left(\frac{4}{3}x^{3/2}-\frac{x^3}{12}\right)_2^4$$

$$= \left(\frac{8\sqrt{2}}{3}-6+2\right) - \left(\frac{4}{3}-3+\frac{1}{2}\right) + \left(\frac{32}{3}-\frac{64}{12}\right) - \left(\frac{8\sqrt{2}}{3}-\frac{8}{12}\right) = \frac{19}{6} \text{ वर्ग इकाई}$$

25. Find the area of the region bounded by y = f(x) , y = | g(x) | and the lines x = 0, x = 2, where ‘f’ , ‘g’ are continuous functions satisfying f(x + y) = f(x) + f(y) – 8xy ∀ x, y ∈ R and g(x + y) = g(x) + g(y) + 3xy (x + y) x, y ∈ R also f’(0) = 8 and g’(0) = – 4.

y = f(x) , y = | g(x) |, x = 0 तथा x = 2 से परिबद्ध क्षेत्रफल ज्ञात कीजिए, जहाँ ‘f’ , ‘g’ सतत् फलन है तथा f(x + y) = f(x) + f(y) – 8xy ∀ x, y ∈ R और g(x + y) = g(x) + g(y) + 3xy (x + y) x, y ∈ R को संतुष्ट करते है। साथ ही f ’(0) = 8 तथा g’(0) = – 4 है।

Ans. $\frac{4}{3}$

Sol. Here f(x + y) = f(x) + f(y) – 8xy.
Replacing x, y by 0 we obtain f(0) = 0

Now, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{y\to 0}\frac{f(x+y)-f(x)}{y} = \lim_{y\to 0}\frac{f(x)+f(y)-8xy-f(x)}{y}$

$= \lim_{y\to 0}\left\{\frac{f(y)}{y}-\frac{8xy}{y}\right\} = f'(0) - 8x$

$= 8 - 8x$ [given $f'(0) = 8$]

$\Rightarrow$ $f'(x) = 8 - 8x$

Integrating both side,

$f(x) = 8x - 4x^2 + c$

as $f(0) = 0$ $\Rightarrow$ $c = 0$

$\Rightarrow$ $f(x) = 8x - 4x^2$

also $g(x+y) = g(x) + g(y) + 3xy\ (x+y)$

Replacing x, y by 0, we obtain $g(0) = 0$

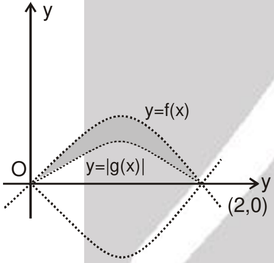
Now $g'(x) = \lim_{y\to 0}\frac{g(x+y)-g(x)}{y}$

$= \lim_{y\to 0}\frac{g(x)+g(y)+3x^2y+3xy^2-g(x)}{y}$

$= \lim_{y\to 0}\left[\frac{g(y)}{y}+\frac{y(3x^2+3xy)}{y}\right]$

$= g'(0) + 3x^2 = -4 + 3x^2$

$\therefore$ $g(x) = x^3 - 4x$ (as $g(0) = 0$) .... (ii)



$$|g(x)| = \begin{cases} x^3 - 4x, & x \in [-2\ ,\ 0] \cup [2,\ \infty) \\ 4x - x^3\ , & x \in (-\infty\ ,\ -2) \cup (0,\ 2) \end{cases}$$

Points where f(x) and |g(x)| meets, we have

$f(x) = |g(x)|$ $\Rightarrow$ $x = 0\ ,\ 2.$

Area bounded by $y = f(x)$ and $y = |g(x)|$,

between $x = 0$ to $x = 2$ is $\int_0^2 (x^3 - 4x^2 + 4x)\ dx = \frac{4}{3}$.

Hindi यहाँ $f(x + y) = f(x) + f(y) - 8xy$

x, y को 0 से प्रतिस्थापित करने पर हमें $f(0) = 0$ प्राप्त होता है।

अब, $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{y\to 0}\frac{f(x+y)-f(x)}{y} = \lim_{y\to 0}\frac{f(x)+f(y)-8xy-f(x)}{y}$

$= \lim_{y\to 0}\left\{\frac{f(y)}{y}-\frac{8xy}{y}\right\} = f'(0) - 8x$

$= 8 - 8x$ [दिया हुआ है $f'(0) = 8$]

$\Rightarrow$ $f'(x) = 8 - 8x$

दोनों तरफ समाकलन करने पर

$f(x) = 8x - 4x^2 + c$

जैसे कि $f(0) = 0$ $\Rightarrow$ $c = 0$ $\Rightarrow$ $f(x) = 8x - 4x^2$

साथ ही $g(x+y) = g(x) + g(y) + 3xy\ (x+y)$

x, y को 0 से प्रतिस्थापित करने पर हमें g(0) = 0 प्राप्त होता है।

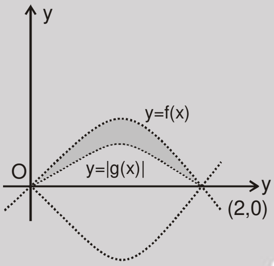
अब $g'(x) = \lim\limits_{y\to 0} \dfrac{g(x+y)-g(x)}{y}$

$= \lim\limits_{y\to 0} \dfrac{g(x)+g(y)+3x^2y+3xy^2-g(x)}{y}$

$= \lim\limits_{y\to 0} \left[\dfrac{g(y)}{y} + \dfrac{y(3x^2+3xy)}{y}\right]$

$= g'(0) + 3x^2 = -4 + 3x^2$

$\therefore\ g(x) = x^3 - 4x$ (जैसे कि g(0) = 0) .... (ii)



$$|g(x)| = \begin{cases} x^3-4x, & x \in [-2\ ,\ 0] \cup [2,\ \infty) \\ 4x-x^3\ , & x \in (-\infty\ ,\ -2) \cup (0,\ 2) \end{cases}$$

बिन्दु, जहाँ f(x) तथा |g(x)| मिलते है

f(x) = |g(x)| $\Rightarrow$ x = 0 , 2.

y = f(x), y = | g(x) |, x = 0 तथा x = 2 से परिबद्ध क्षेत्रफल $\int_0^2 (x^3-4x^2+4x)\ dx = \dfrac{4}{3}$.

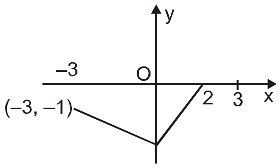
26. Let $f(x) = \begin{cases} -2 & , & -3 \le x \le 0 \\ x-2 & , & 0 < x \le 3 \end{cases}$, where g(x) = min {f(|x|) + |f(x)|, f(|x|) – |f(x)|}

Find the area bounded by the curve g(x) and the x-axis between the ordinates x = 3 and x = –3.

माना $f(x) = \begin{cases} -2 & , & -3 \le x \le 0 \\ x-2 & , & 0 < x \le 3 \end{cases}$, जहाँ g(x) = न्यूनतम {f(|x|) + |f(x)|, f(|x|) – |f(x)|}

वक्र g(x), x-अक्ष तथा कोटियें x = 3 और x = –3 के मध्य परिबद्ध क्षेत्र का क्षेत्रफल ज्ञात कीजिए।

Ans. $\dfrac{23}{2}$



Graph of y = g(x)

Sol. Here $f(|x|) = \begin{cases} -x-2 & , & -3 \le x \le 0 \\ x-2 & , & 0 < x \le 3 \end{cases}$

$$|f(x)| = \begin{cases} 2 & , & -3 \le x \le 0 \\ -x+2 & , & 0 < x \le 2 \\ x-2 & , & 2 < x \le 3 \end{cases}$$

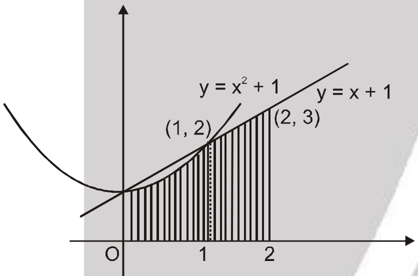
$$= \left.\frac{x^3}{3}+x\right|_0^1 + \left.\frac{x^2}{2}+x\right|_1^2$$

$$= \frac{23}{6} \text{ sq. units}$$

Hindi माना $R = \{(x,\ y) : 0 \le y \le x^2+1,\ \ 0 \le y \le x+1,\ \ 0 \le x\ \ \le 2\}$.

$= \{(x,\ y) : 0 \le y \le x^2+1\}\ \cap \{(x,y) : 0 \le y \le x+1\} \cap\ \{(x,y) :\ 0 \le x\ \ \le 2\}$

इसलिए अभीष्ट क्षेत्रफल



चित्र

$$= \int_0^1 (x^2+1)\ \ dx + \int_1^2 (x+1)\ \ dx$$

$$= \left.\frac{x^3}{3}+x\right|_0^1 + \left.\frac{x^2}{2}+x\right|_1^2$$

$= \frac{23}{6}$ वर्ग इकाई

28. A curve $y = f(x)$ passes through the point $P(1, 1)$, the normal to the curve at P is $a(y - 1) + (x - 1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of that point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P.

एक वक्र $y = f(x)$ बिन्दु $P(1, 1)$ से गुजरता है तथा P पर अभिलम्ब $a(y - 1) + (x - 1) = 0$ है। यदि वक्र के किसी बिन्दु पर स्पर्श रेखा की प्रवणता उस बिन्दु की कोटि के समानुपाती है तो वक्र का समीकरण ज्ञात कीजिए। साथ ही y-अक्ष , वक्र तथा P पर अभिलम्ब से परिबद्ध क्षेत्र का क्षेत्रफल ज्ञात कीजिए।

Ans. $y = e^{a(x-1)}$, $\left(1+\frac{e^{-a}}{a}-\frac{1}{2a}\right)$

Sol. Here, slope of the normal at $P(x, y)$

$\Rightarrow$ slope of the line $a(y - 1) + (x - 1) = 0$

$= -\frac{1}{a}$

$\therefore$ slope of tangent = a

Now $\left(\frac{dy}{dx}\right)_{(1,\ 1)} = \lambda$

$\therefore$ $a = \lambda$

$\Rightarrow \frac{dy}{dx} = ay$

$\Rightarrow$ $y = Ce^{ax}$ this passes through (1, 1)

$\therefore$ equation of curve is

$y = e^{-a} . e^{ax} \Rightarrow y = e^{a(x-1)}$

$\therefore$ required area = $\frac{1}{a}\int_{e^{-a}}^{1}(\log y + a)\ dy + \int_{1}^{1+\frac{1}{a}}(1+a-ay)dy$

Area = $\left(1+\frac{e^{-a}}{a}-\frac{1}{2a}\right)$

Alternative :

Area = $\int_{0}^{1}\left(1+\frac{1}{a}-\frac{x}{a}-e^{a(x-1)}\right)\ dx\ = \left(1+\frac{e^{-a}}{a}-\frac{1}{2a}\right)$

Hindi P(x, y) पर अभिलम्ब की प्रवणता

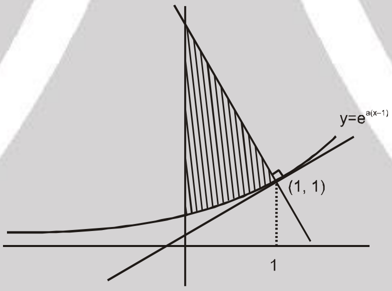
$\Rightarrow$ रेखा a(y – 1) + (x – 1) = 0 की प्रवणता

$= -\frac{1}{a}$

$\therefore$ स्पर्श रेखा की प्रवणता = a

अब $\left(\frac{dy}{dx}\right)_{(1,\ 1)} = \lambda$

$\therefore$ a = λ



चित्र

$\Rightarrow \frac{dy}{dx} = ay$

$\Rightarrow$ y = Ce^{ax} यह (1, 1) से गुजरता है।

$\therefore$ वक्र का समीकरण है

$y = e^{-a} . e^{ax} \Rightarrow y = e^{a(x-1)}$

$\therefore$ अभीष्ट क्षेत्रफल = $\frac{1}{a}\int_{e^{-a}}^{1}(\log y + a)\ dy + \int_{1}^{1+\frac{1}{a}}(1+a-ay)dy$

हल करने पर क्षेत्रफल = $\left(1+\frac{e^{-a}}{a}-\frac{1}{2a}\right)$

वैकल्पिक :

क्षेत्रफल = $\int_{0}^{1}\left(1+\frac{1}{a}-\frac{x}{a}-e^{a(x-1)}\right)\ dx\ = \left(1+\frac{e^{-a}}{a}-\frac{1}{2a}\right)$

29. Find the area bounded by y = [– 0. 01 x^4 – 0.02 x^2], (where [.] G.I.F.) and curve 3x^2 + 4y^2 = 12, which lies below y = – 1.

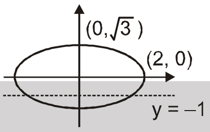
y = [–0. 01 x^4 – 0.02 x^2], (जहाँ [.] महत्तम पूर्णांक फलन है) तथा वक्र 3x^2 + 4y^2 = 12 से परिबद्ध क्षेत्र का क्षेत्रफल ज्ञात कीजिए जो y = – 1 के नीचे स्थित है।

Ans. $2\ \sqrt{3}\sin^{-1}\sqrt{\frac{2}{3}}-\frac{2\sqrt{2}}{\sqrt{3}}$

Sol. Here $y = [-0.01x^4 - 0.02x^2]$

$y = -1$ when $-2 < x < 2$

$y = -1$ cuts the ellipse $3x^2 + 4y^2 = 12$



Figure

at $x = \frac{\pm 2\sqrt{2}}{\sqrt{3}}$

Required area

$$= \int_{\frac{-2\sqrt{6}}{3}}^{\frac{2\sqrt{6}}{3}} \left(\sqrt{\frac{12-3x^2}{4}} - 1\right) dx$$

$$= 2\sqrt{3}\sin^{-1}\sqrt{\frac{2}{3}} - \frac{2\sqrt{2}}{\sqrt{3}}$$

Hindi यहाँ $y = [-0.01x^4 - 0.02x^2]$

$y = -1$ जब $-2 < x < 2$

$y = -1$ दीर्घवृत्त $3x^2 + 4y^2 = 12$ को $x = \frac{\pm 2\sqrt{2}}{3}$ पर काटता है

अभीष्ट क्षेत्रफल

$$= \int_{\frac{-2\sqrt{6}}{3}}^{\frac{2\sqrt{6}}{3}} \left(\sqrt{\frac{12-3x^2}{4}} - 1\right) dx$$

$$= 2\sqrt{3}\sin^{-1}\sqrt{\frac{2}{3}} - \frac{2\sqrt{2}}{\sqrt{3}}$$

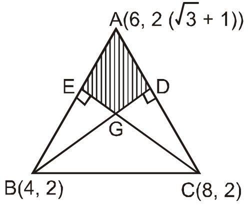
30. Let ABC be a triangle with vertices $A(6, 2(\sqrt{3} + 1))$, $B(4, 2)$ and $C(8, 2)$. If R be the region consisting of all these points and point P inside ΔABC which satisfy $d(P, BC) \geq \max. \{d(P, AB), d(P, AC)\}$ where $d(P, L)$ denotes the distance of the point P from the line L. Sketch the region R and find its area.

माना ABC एक त्रिभुज है जिसके शीर्ष $A(6, 2(\sqrt{3} + 1))$, $B(4, 2)$ तथा $C(8, 2)$ है। यदि R इन सभी बिन्दुओं को समाहित करने वाले क्षेत्र को प्रदर्शित करे तथा बिन्दु P, ΔABC के अन्दर इस प्रकार स्थित है कि $d(P, BC) \geq \max. \{d(P, AB), d(P, AC)\}$ जहाँ $d(P, L)$ बिन्दु P की रेखा L से दूरी प्रदर्शित करता है। क्षेत्र R का रेखाचित्र खींचिए तथा इसका क्षेत्रफल ज्ञात कीजिए।

Ans. $\frac{4\sqrt{3}}{3}$

Sol. As the given triangle is equilateral with side lengths 4. BD and CE are angle bisectors of angle B and C resp. Any point inside the ΔAEC is nearer to AC than BC and any point inside the ΔBDA is nearer to AB than BC. So points inside the quadrilateral AEGD will satisfy the given condition

$\therefore$ Required area = 2 (ΔEAG)



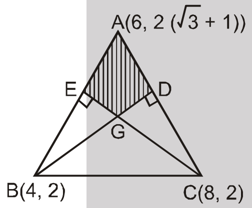
$$= \quad 2 \times \frac{1}{2} \times AE \times EG$$

$= \dfrac{4\sqrt{3}}{3}$ sq. units.

Hindi. जैसे कि दिया गया त्रिभुज समबाहु है जिसकी भुजाओं की लम्बाई 4 है। कोण B तथा C के कोण अर्द्धक क्रमशः BD तथा CE है। ΔAEC में कोई बिन्दु BC की तुलना में AC से ज्यादा नजदीक है तथा ΔABDA में कोई बिन्दु BC की तुलना में AB के ज्यादा नजदीक है।

इसलिए चतुर्भुज AEGD के अन्दर स्थित बिन्दु दिये गये प्रतिबन्ध को संतुष्ट करेंगे।

A(6, 2 ($\sqrt{3}$ + 1))



$\therefore$ अभीष्ट क्षेत्रफल = 2 (ΔEAG)

$= 2 \times \dfrac{1}{2} \times AE \times EG$

$= \dfrac{4\sqrt{3}}{3}$ वर्ग इकाई

31. Find the area of the region which contains all the points satisfying the condition $|x - 2y| + |x + 2y| \le 8$ and $xy \ge 2$.

उस क्षेत्र का क्षेत्रफल ज्ञात कीजिए जिसमें वे सभी बिन्दु स्थित है जो प्रतिबन्ध $|x - 2y| + |x + 2y| \le 8$ तथा $xy \ge 2$ को संतुष्ट करें।

Ans. 2(6 – 2 log 4)

Sol. The lines $y = \pm \dfrac{x}{2}$ divides the xy – plane in four part

Region I $2y - x \le 0$ and $2y + x \ge 0$. $\therefore$ $(x - 2y) + (x + 2y) \le 8$.
$0 \le x \le 4$.

Region II $2y - x \ge 0$ and $2y + x \ge 0$
$-(x - 2y) + (x + 2y) \le 0$
$-(x - 2y) + (x + 2y) \le 8$
$0 \le y \le 2$

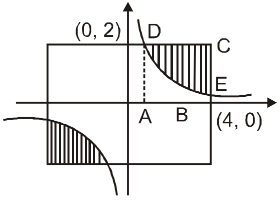
Region III $2y + x \le 0,\ 2y - x \ge 0$
$-4 \le x \le 0$

Region IV $2y + x \le 0,\ 2y - x \le 0$
$-2 \le y \le 0$

Here all points lie in the rectangle.

Also the hyperbola xy = 2 meets the sides of the rectangle at the points (1, 2) and (4, 1/2) in the Ist quadrant and similarly at two points in 3rd quadrant as in figure.

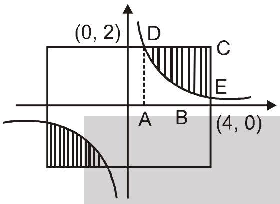
Hence the required area



Figure

= 2((area of rectangle ABCD) – (area of ABEDA))

$$= 2\left(3\times2-\int_1^4 \frac{2}{x}\ dx\right) = 2(6 - 2\log 4)$$



चित्र

Hindi रेखा $y = \frac{x}{2} \pm$, xy – समतल को चार भागों में विभाजित करता है।

क्षेत्र I $2y - x \le 0$ and $2y + x \ge 0$. $\therefore$ $(x - 2y) + (x + 2y) \le 8$.

$0 \le x \le 4$.

क्षेत्र II $2y - x \ge 0$ and $2y + x \ge 0$

$-(x - 2y) + (x + 2y) \le 0$

$-(x - 2y) + (x + 2y) \le 8$

$0 \le y \le 2$

क्षेत्र III $2y + x \le 0,\ 2y - x \ge 0$

$-4 \le x \le 0$

क्षेत्र IV $2y + x \le 0,\ 2y - x \le 0$

$-2 \le y \le 0$

यहाँ सभी बिन्दु आयत के अन्दर स्थित है

साथ ही अतिपरवलय xy = 2 प्रथम चतुर्थांश में आयत की भुजाओं से बिन्दुओं (1, 2) तथा (4, 1/2) पर मिलता है तथा इसी प्रकार तृतीय चतुर्थांश में दो बिन्दुओं पर चित्र में दर्शाये अनुसार मिलता है।

अतः अभीष्ट क्षेत्रफल

= 2((आयत ABCD का क्षेत्रफल) – (ABEDA का क्षेत्रफल))

$$= 2\left(3\times2-\int_1^4 \frac{2}{x}\ dx\right) = 2(6 - 2\log 4)$$

32. Find the area of the region which is inside the parabola $y = -x^2 + 6x - 5$, out side the parabola $y = -x^2 + 4x - 3$ and left of the straight line $y = 3x - 15$.

उस क्षेत्र का क्षेत्रफल ज्ञात कीजिए जो परवलय $y = -x^2 + 6x - 5$ के अंदर, परवलय $y = -x^2 + 4x - 3$ के बाहर तथा सरल रेखा $y = 3x - 15$ के बाँयी तरफ स्थित है।

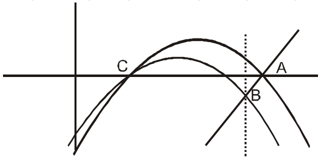
Ans. $\frac{73}{6}$

Sol. $y = -(x^2 - 6x + 5) = -(x - 5)(x - 1)$

$y = -(x^2 - 4x + 3) = -(x - 3)(x - 1)$

$y = 3x - 15$

A (5, -0) B(4, – 3) C (1, 0).



Figure

Area क्षेत्रफल $= \int_1^4 \left(\left(-x^2+6x-5\right)-\left(-x^2+4x-3\right)\right)dx + \int_4^5 \left(\left(-x^2+6x-5\right)-\left(3x-15\right)\right)dx$

$= \int_1^5 (-x^2+6x-5)dx - \int_1^4 (-x^2+4x-3)dx - \int_4^5 (3x-15)dx$

$= \left(-\frac{x^3}{3}+3x^2-5x\right)_1^5 - \left(-\frac{x^3}{3}+2x^2-3x\right)_1^4 - \left(\frac{3x^2}{2}-15x\right)_4^5 = \frac{32}{3} - (0) + \frac{3}{2}$

Area क्षेत्रफल $= \frac{73}{6}$

33. Consider the curve C: $y = \sin 2x - \sqrt{3}\ |\sin x|$, C cuts the x − axis at (a, 0), $x \in (-\pi, \pi)$.

A_1 : The area bounded by the curve C and the positive x − axis between the origin and the line x = a.

A_2 : The area bounded by the curve C and the negative x − axis between the line x = a and the origin.

Prove that $A_1 + A_2 + 8\ A_1 A_2 = 4$.

वक्र C: $y = \sin 2x - \sqrt{3}\ |\sin x|$, x−अक्ष को बिन्दु (a, 0), $x \in (-\pi, \pi)$ पर काटता है।

A_1: वक्र C तथा धनात्मक x−अक्ष द्वारा मूल बिन्दु और x = a के मध्य परिबद्ध क्षेत्र का क्षेत्रफल

A_2: वक्र C तथा ऋणात्मक x−अक्ष द्वारा मूल बिन्दु और x = a के मध्य परिबद्ध क्षेत्र का क्षेत्रफल

सिद्ध कीजिए कि $A_1 + A_2 + 8\ A_1 A_2 = 4$

Sol. $y = \sin 2x - \sqrt{3}\ |\sin x|$

for $-\pi < x \le 0$ $\quad y_2 = 2\sin x\left(\cos x + \frac{\sqrt{3}}{2}\right)$

for $0 \le x < \pi$ $\quad y_1 = 2\sin x\left(\cos x - \frac{\sqrt{3}}{2}\right)$

(i) $y_1 = 0$ at x = 0 and $\frac{\pi}{6}$

$$A_1 = \int_0^{\pi/6} (\sin 2x - \sqrt{3}\sin x)\ \ dx$$

$$= -\frac{\cos 2x}{2} + \sqrt{3}\cos x \Big|_0^{\pi/6}$$

$$= -\frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}\ .\ \sqrt{3}}{2} + \frac{1}{2} - \sqrt{3}$$

$$= \frac{7}{4} - \sqrt{3}$$

(ii) $y_2 = 0$ at x = 0 and $x = -\frac{5\pi}{6}$

$$A_2 = \int_{-\frac{5\pi}{6}}^{0} (\sin 2x + \sqrt{3}\sin x)dx = \left|\int_{-\frac{5\pi}{6}}^{0} (\sin 2x + \sqrt{3}\sin x)dx\right|_{-\frac{5\pi}{6}}^{0}$$

$$= \left|-\frac{1}{2} - \sqrt{3} + \frac{1}{2}\ .\ \frac{1}{2} + \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)\right| \quad = \quad \text{or} \quad A_2 = \frac{7}{4} + \sqrt{3}$$

$$A_1 + A_2 + 8A_1A_2 = \frac{7}{4} - \sqrt{3} + \frac{7}{4} + \sqrt{3} + 8\left(\frac{49}{16} - 3\right) = \frac{7}{2} + \frac{49}{2} - 24 = 4$$

Sol. $y = \sin 2x - \sqrt{3}\ |\sin x|$

$-\pi < x \le 0$ के लिए $\qquad y_2 = 2 \sin x \left(\cos x + \frac{\sqrt{3}}{2}\right)$

$0 \le x < \pi$ के लिए $\qquad y_1 = 2 \sin x \left(\cos x - \frac{\sqrt{3}}{2}\right)$

(i) $x = 0$ तथा $\frac{\pi}{6}$ पर $y_1 = 0$

$$A_1 = \int_0^{\pi/6} (\sin 2x - \sqrt{3}\sin x)\ dx$$

$$= -\frac{\cos 2x}{2} + \sqrt{3}\cos x \Big|_0^{\pi/6}$$

$$= -\frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}\ .\ \sqrt{3}}{2} + \frac{1}{2} - \sqrt{3}$$

$$= \frac{7}{4} - \sqrt{3}$$

(ii) $x = 0$ तथा $x = -\frac{5\pi}{6}$ पर $y_2 = 0$

$$A_2 = \left|\int_{-\frac{5\pi}{6}}^{0} (\sin 2x + \sqrt{3}\sin x)\ dx\right|$$

$$= \left|-\frac{1}{2} - \sqrt{3} + \frac{1}{2}\ .\ \frac{1}{2} + \sqrt{3}\left(-\frac{\sqrt{3}}{2}\right)\right| \quad = \quad \text{or} \quad A_2 = \frac{7}{4} + \sqrt{3}$$

$$A_1 + A_2 + 8A_1 A_2 = \frac{7}{4} - \sqrt{3} + \frac{7}{4} + \sqrt{3} + 8\left(\frac{49}{16} - 3\right) = \frac{7}{2} + \frac{49}{2} - 24 = 4$$

34. Area bounded by the line y = x, curve y = f(x), (f(x) > x ∀ x > 1) and the lines x = 1, x = t is $(t + \sqrt{1+t^2} - (1 + \sqrt{2})\ \forall\ t > 1$. Find f(x) for x > 1.

रेखा y = x , वक्र y = f(x), (f(x) > x ∀ x > 1) तथा रेखाओं x = 1, x = t से परिबद्ध क्षेत्रफल $(t + \sqrt{1+t^2} - (1 + \sqrt{2}\ )\ \forall\ t > 1$ है, तो f(x), x > 1 के लिए ज्ञात कीजिए।

Ans. $1 + x + \frac{x}{\sqrt{1+x^2}}$.

Sol. The area bounded by y = f(x) and y = x between the lines x = 1 and x = t, is

$$\int_1^t (f(x) - x)\ \ dx$$

but it is equal to $(t + \sqrt{1+t^2}) - (1 + \sqrt{2})$

So $\int_1^t (f(x) - x)\ \ dx = (t + \sqrt{1+t^2}) - (1 + \sqrt{2})$

Differentiating both sides w.r.t. t, we get

$$f(t) - t = 1 + \frac{1}{\sqrt{1+t^2}}$$

$$\therefore\quad f(x) = 1 + x + \frac{x}{\sqrt{1+x^2}}.$$

Hindi $y = f(x)$, $y = x$ तथा रेखाओं $x = 1$ और $x = t$ से परिबद्ध क्षेत्रफल है

$$\int_1^t (f(x)-x)\ dx$$

लेकिन यह $(t+\sqrt{1+t^2})-(1+\sqrt{2})$ के बराबर है।

इसलिए $\int_1^t (f(x)-x)\ dx = (t+\sqrt{1+t^2})-(1+\sqrt{2})$

दोनों तरफ t के सापेक्ष अवकलन करने पर

$$f(t) - t = 1 + \frac{1}{\sqrt{1+t^2}}$$

$$\therefore \quad f(x) = 1 + x + \frac{x}{\sqrt{1+x^2}}.$$

35. Consider the two curves $y = 1/x^2$ and $y = 1/[4\,(x-1)]$.

(i) At what value of ‘a’ ($a > 2$) is the reciprocal of the area of the figure bounded by the curves, the lines $x = 2$ and $x = a$ equal to ‘a’ itself ?

(ii) At what value of ‘b’ ($1 < b < 2$) the area of the figure bounded by these curves, the lines $x = b$ and $x = 2$ equal to $1 - 1/b$.

दिये गये दो वक्रों $y = 1/x^2$ तथा $y = 1/[4\,(x-1)]$ के लिए –

(i) ‘a’ के किस मान पर वक्रों और रेखाओं $x = 2$ तथा $x = a$ द्वारा परिबद्ध क्षेत्रफल का व्युत्क्रम ‘a’ के बराबर होगा।

(ii) ‘b’ ($1 < b < 2$) के किस मान पर वक्रों और रेखाओं $x = b$ तथा $x = 2$ द्वारा परिबद्ध क्षेत्रफल $1 - 1/b$ के बराबर होगा।

Ans. (i) $a = 1 + e^2$ (ii) $b = 1 + e^{-2}$

Sol. $y_1 = \frac{1}{x^2}$ and $y_2 = \frac{1}{4(x-1)}$.

These curves are touching at $x = 2$.

$$A = \int_2^a \left(-\frac{1}{x^2} + \frac{1}{4(x-1)}\right) dx = \frac{1}{x} + \frac{1}{4}\ \ell n(x-1)\Big|_2^a$$

or $\left(\frac{1}{a} + \frac{1}{4}\ell n\ (a-1) - \frac{1}{2}\right) = \frac{1}{a}$

or $\ell n\ (a-1) - 2 = 0$

$\Rightarrow$ $a = 1 + e^2$

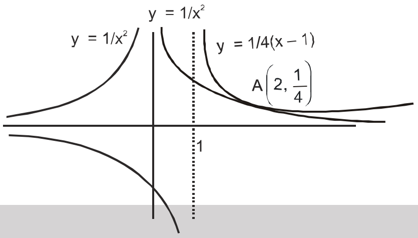
(ii) $A = \int_b^2 \left(-\frac{1}{x^2} + \frac{1}{4(x-1)}\right) dx = 1 - \frac{1}{b}$

or $\frac{1}{2} - \frac{1}{4}\ \ell n\ (b-1) - \frac{1}{b} = 1 - \frac{1}{b}$

or $b = 1 + 1/e^2$

Hindi $y_1 = \frac{1}{x^2}$ तथा $y_2 = \frac{1}{4(x-1)}$.

ये वक्र $x = 2$ पर स्पर्श करते है।



चित्र

$$A = \int_2^a \left(-\frac{1}{x^2} + \frac{1}{4(x-1)}\right) dx \; = \; \frac{1}{x} \; + \; \frac{1}{4} \quad \ell n(x-1)\Big|_2^a$$

या $\left(\frac{1}{a} + \frac{1}{4}\ell n \;\; (a-1) - \frac{1}{2}\right) \; = \; \frac{1}{a}$

या $\ell n\ (a - 1) - 2 = 0$

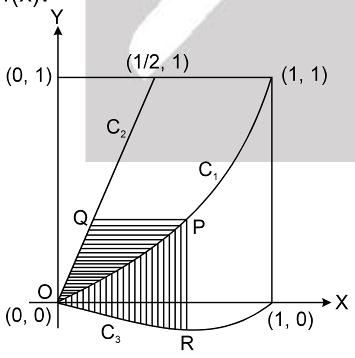
$\Rightarrow$ $a = 1 + e^2$

(ii) $A = \int_b^2 \left(-\frac{1}{x^2} + \frac{1}{4(x-1)}\right) dx \; = 1 - \frac{1}{b}$

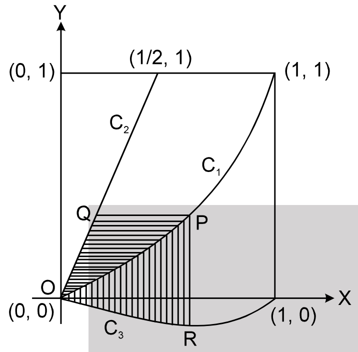
या $\frac{1}{2} - \frac{1}{4} \; \ell n\ (b - 1) - \frac{1}{b} \; = 1 - \frac{1}{b}$

या $b = 1 + 1/e^2$

36. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function f(x).



माना कि C_1 तथा C_2 क्रमशः फलन $y = x^2$ तथा $y = 2x$ जबकि $0 \le x \le 1$ के रेखाचित्र हैं। माना कि C_3 फलन $y = f(x)$, $0 \le x \le 1$, का रेखाचित्र है, $f(0)=0$, C_1 पर स्थित किसी बिन्दु P के लिए माना कि बिन्दु P से गुजरने वाली तथा अक्षों के समान्तर रेखाऐं C_2 तथा C_3 को क्रमशः Q तथा R पर चित्रानुसार मिलती है। यदि C_1 पर स्थित बिन्दु P की प्रत्येक स्थिति के लिए छायांकित क्षेत्र OPQ एवं ORP के क्षेत्रफल बराबर हो, तो फलन f(x) ज्ञात कीजिए।



Ans. $f(x) = x^3 - x^2$

Sol. Let the coordinates of P be (x,x^2), where $0 \le x \le 1$. Area (OPRO) = Area (OPQO)

$$\Rightarrow \int_0^x t^2 dt - \int_0^x f(t)\ dt = \int_0^{x^2} \sqrt{t}\ dt - \int_0^{x^2} \frac{t}{2}\ dt \Rightarrow \frac{1}{3}x^3 - \int_0^x f(t)\ dt = \frac{2}{3}x^3 - \frac{x^4}{4}$$

Differentiating both sides w.r.t.x, we get

$x^2 - f(x)\ .\ 1 = 2x^2 - x^3$

$\Rightarrow$ $f(x) = x^3 - x^2,\ 0 \le x \le 1$

Hindi. माना P के निर्देशांक (x,x^2) है, जहाँ $0 \le x \le 1$

क्षेत्रफल (OPRO) = क्षेत्रफल (OPQO)

$$\Rightarrow \int_0^x t^2 dt - \int_0^x f(t)\ dt = \int_0^{x^2} \sqrt{t}\ dt - \int_0^{x^2} \frac{t}{2}\ dt \Rightarrow \frac{1}{3}x^3 - \int_0^x f(t)\ dt = \frac{2}{3}x^3 - \frac{x^4}{4}$$

दोनों तरफ x के सापेक्ष अवकलन करने पर

$x^2 - f(x)\ .\ 1 = 2x^2 - x^3 \Rightarrow$ $f(x) = x^3 - x^2,\ 0 \le x \le 1$

37. Given the parabola C : $y = x^2$. If the circle centred at y axis with radius 1 touches parabola C at two distinct points, then find the coordinate of the center of the circle K and the area of the figure surrounded by C and K.

दिया गया है कि परवलय C : $y = x^2$ यदि वृत्त का केन्द्र y-अक्ष पर तथा त्रिज्या 1 है, परवलय को दो विभिन्न बिन्दुओं पर स्पर्श करती है। तब वृत्त K के निर्देशांक ज्ञात कीजिए तथा C और K से परिबद्ध क्षेत्र का क्षेत्रफल ज्ञात कीजिए।

Ans. centre $\left(0, \frac{5}{4}\right)$ and area $= \frac{3\sqrt{3}}{4} - \frac{\pi}{3}$

केन्द्र $\left(0, \frac{5}{4}\right)$ और क्षेत्रफल $= \frac{3\sqrt{3}}{4} - \frac{\pi}{3}$

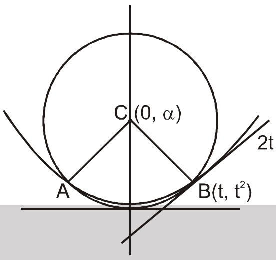
Sol. Let C be centre of circle & A, B be points of contact. If B is (t, t^2) $(t > 0)$

$$\left.\frac{dy}{dx}\right|_B = 2t = \frac{t}{\alpha - t^2} \quad \Rightarrow \quad \alpha - t^2 = \frac{1}{2} \Rightarrow t^2 = \alpha - \frac{1}{2}$$

Now $CB^2 = 1 \Rightarrow t^2 + (t^2 - \alpha)^2 = 1$

$\Rightarrow$ Hence centre of circle is $\left(0, \frac{5}{4}\right)$ & point of contacts are $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$

Hence area $= 2\displaystyle\int_0^{\frac{\sqrt{3}}{2}} \left\{\frac{5}{4} - \sqrt{1 - x^2} - x^2\right\} dx$



$$= 2\left[\frac{5x}{4} - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}(x) - \frac{x^3}{3}\right]_0^{\sqrt{3}/2}$$

$$= 2\left[\frac{5}{4}.\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}.\frac{1}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{8}\right] = \frac{3\sqrt{3}}{4} - \frac{\pi}{3}$$

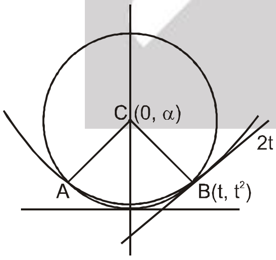
Hindi. माना कि वृत्त का केन्द्र A और B स्पर्श बिन्दु है यदि B (t, t²) है (t > 0)

$$\left.\frac{dy}{dx}\right|_B = 2t = \frac{t}{\alpha - t^2} \quad \Rightarrow \quad \alpha - t^2 = \frac{1}{2} \Rightarrow t^2 = \alpha - \frac{1}{2}$$

अब $CB^2 = 1 \Rightarrow \quad t^2 + (t^2 - \alpha)^2 = 1$

$\Rightarrow$ अतः वृत्त का केन्द्र $\left(0, \frac{5}{4}\right)$ तथा स्पर्श बिन्दु $\left(\frac{\sqrt{3}}{2}, \frac{3}{4}\right)$

अतः क्षेत्रफल $= 2\int_0^{\frac{\sqrt{3}}{2}} \left\{\frac{5}{4} - \sqrt{1-x^2} - x^2\right\}dx$



$$= 2\left[\frac{5x}{4} - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}(x) - \frac{x^3}{3}\right]_0^{\sqrt{3}/2}$$

$$= 2\left[\frac{5}{4}.\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}.\frac{1}{2} - \frac{\pi}{3} - \frac{\sqrt{3}}{8}\right] = \frac{3\sqrt{3}}{4} - \frac{\pi}{3}$$

38. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, f(x) is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of y = f(x) with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by f(x) and chord AB.

यदि $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, एक द्विघात फलन हैं तथा एक बिन्दु V पर इसका अधिकतम मान प्राप्त होता है। y = f(x) का x-अक्ष के साथ प्रतिच्छेद बिन्दु A हैं तथा बिन्दु B इस प्रकार है कि जीवा AB बिन्दु V पर समकोण बनाती है। f(x) तथा जीवा AB से परिबद्ध क्षेत्रफल ज्ञात कीजिए।

Ans. $\frac{125}{3}$ square units. (वर्ग इकाई)

Sol. From given equation

$\Rightarrow$ $4a^2 f(-1) + 4a\, f(1) + f(2) = 3a^2 + a$ ....(1)

$4b^2 f(-1) + 4b\, f(1) + f(2) = 3b^2 + b$ ....(2)

$4c^2 f(-1) + 4c\, f(1) + f(2) = 3c^2 + c$ ....(3)

where f(x) is quadratic expression given by,

$f(x) = ax^2 + bx + c$ and (1), (2) and (3)

$\Rightarrow$ $4x^2 f(-1) + 4x\, f(1) + f(2) = 3x^2 + 3x$

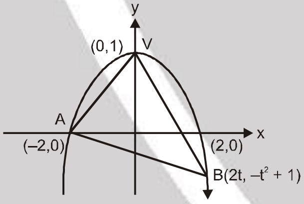
or $(4f(-1) - 3)x^2 + (4f(1) - 3)x + f(2) = 0$ ....(4)

As above equation has 3 roots a, b and c.

above equation is identity in x.

i.e., coefficients must be zero

$\Rightarrow$ $f(-1) = 3/4,\ f(1) = 3/4,\ f(2) = 0$ ....(5)



$f(x) = ax^2 + bx + c$

$\Rightarrow$ $a = -1/4,\ b = 0$ and $c = 1$, using (5)

Thus, $f(x) = \frac{4-x^2}{4}$ shown as,

Let A (–2, 0), B = (2t, – t² + 1)

Since AB subtends right angle at vertex V (0, 1)

$\Rightarrow \frac{1}{2}\cdot\frac{-t^2}{2t} = -1 \qquad \Rightarrow \qquad t = 4$

B(8, –15)

Thus, Area = $\int_{-2}^{8}\left(\frac{4-x^2}{4} + \frac{3x+6}{2}\right) dx$

$= \left(x - \frac{x^3}{12} + \frac{3x^2}{4} + 3x\right)_{-2}^{8} = \frac{125}{3}$ sq. units

Hindi दिये गये समीकरण से

$\Rightarrow$ $4a^2 f(-1) + 4a\, f(1) + f(2) = 3a^2 + a$ ....(1)

$4b^2 f(-1) + 4b\, f(1) + f(2) = 3b^2 + b$ ....(2)

$4c^2 f(-1) + 4c\, f(1) + f(2) = 3c^2 + c$ ....(3)

जहाँ f(x) द्विघात व्यंजक है,

$f(x) = ax^2 + bx + c$, (1), (2) तथा (3) से प्राप्त होता है

$\Rightarrow$ $4x^2 f(-1) + 4x\, f(1) + f(2) = 3x^2 + 3x$

या $(4f(-1) - 3)x^2 + (4f(1) - 3)x + f(2) = 0$ ....(4)

जैसा कि उपरोक्त समीकरण के तीन मूल a, b तथा c है

उपरोक्त समीकरण x में सर्वसमिका है

अर्थात् गुणांक शून्य के बराबर होने चाहिए

$\Rightarrow$ f(–1) = 3/4, f(1) = 3/4, f(2) = 0 ....(5)

f(x) = ax² + bx + c $\Rightarrow$ a = –1/4, b = 0 तथा c = 1, (5) के प्रयोग से

अतः $f(x) = \dfrac{4-x^2}{4}$ का आलेख

माना A (–2, 0), B = (2t, – t² + 1)

$\because$ AB शीर्ष V(0, 1) पर समकोण बनाती है

$$\Rightarrow \frac{1}{2}\cdot\frac{-t^2}{2t} = -1 \Rightarrow \qquad t = 4$$

B(8, –15)

$$\text{अतः क्षेत्रफल} = \int_{-2}^{8}\left(\frac{4-x^2}{4}+\frac{3x+6}{2}\right)dx$$

$$= \left(x-\frac{x^3}{12}+\frac{3x^2}{4}+3x\right)_{-2}^{8} = \frac{125}{3} \text{ वर्ग इकाई}$$

39_. f(x) and g(x) are polynomials of degree 2 such that $\left|\int_{a_1}^{a_2}(f(x)-1)dx\right| = \left|\int_{b_1}^{b_2}(g(x)-1)dx\right|$

where a_1, a_2 ($a_2 > a_1$) are roots of equation f(x) = 1 and b_1, b_2 ($b_2 > b_1$) are roots of equation g(x) = 1. If f″(x) and g″(x) are positive constant and

$$\left|\int_{a_1}^{a_2} f(x))dx\right| = (a_2 - a_1) - \left|\int_{a_1}^{a_2}(f(x)-1)dx\right| \text{ but } \left|\int_{b_1}^{b_2} g(x))dx\right| \neq (b_2-b_1) - \left|\int_{b_1}^{b_2}(g(x)-1)dx\right| \text{ then}$$

(A*) |f″(x)| < |g″(x)| (B) |f″(x)| > |g″(x)| (C*) $a_2 - a_1 > b_2 - b_1$ (D) $a_2 - a_1 > b_2 - b_1$

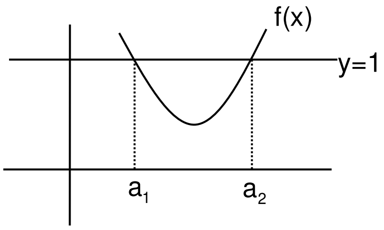
दो घात के बहुपद f(x) और g(x) इस प्रकार है कि $\left|\int_{a_1}^{a_2}(f(x)-1)dx\right| = \left|\int_{b_1}^{b_2}(g(x)-1)dx\right|$ जहां a_1, a_2 ($a_2 > a_1$) समीकरण f(x) = 1 तथा b_1, b_2 ($b_2 > b_1$) समीकरण g(x) = 1 के मूल है यदि f″(x) और g″(x) धनात्मक अचर है

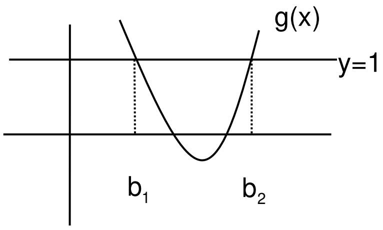
$$\left|\int_{a_1}^{a_2} f(x))dx\right| = (a_2 - a_1) - \left|\int_{a_1}^{a_2}(f(x)-1)dx\right| \text{ परन्तु } \left|\int_{b_1}^{b_2} g(x))dx\right| \neq (b_2-b_1) - \left|\int_{b_1}^{b_2}(g(x)-1)dx\right| \text{ तब}$$

(A*) |f″(x)| < |g″(x)| (B) |f″(x)| > |g″(x)| (C*) $a_2 - a_1 > b_2 - b_1$ (D) $a_2 - a_1 > b_2 - b_1$

Sol. Area between y = f(x) and y = 1 is equal to area between y = g(x) and y = 1

Here minmimum value of f(x) is greater than zero and minimum value of g(x) is less than zero

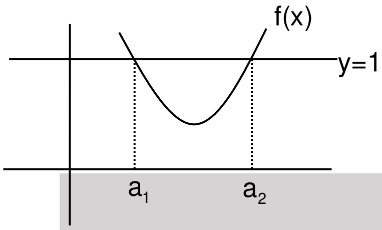


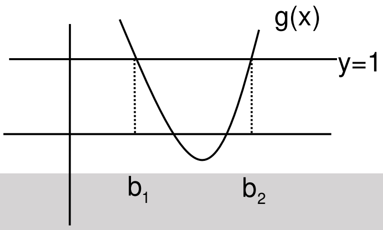


$\Rightarrow$ coefficient x² in g(x) is greater than coefficient of x² in f(x) and difference of root of g(x) = 1 is less then difference of root of f(x) = 1

$\Rightarrow$ $(a_2 - a_1) > (b_2 - b_1)$

Hindi. वक्र y = f(x) और y = 1 के मध्य क्षेत्र का क्षेत्रफल, y = g(x) और y = 1 के मध्य के क्षेत्रफल के बराबर है। यहाँ f(x) का न्यूनतम, 0 से बड़ा है तथा g(x) का न्यूनतम मान, शून्य से कम है।





⇒ g(x) मे x^2 का गुणांक f(x) मे x^2 के गुणांक से बड़ा है तथा g(x) = 1 के मूलों का अन्तर, f(x) = 1 के मूलों के अन्तर से कम है।

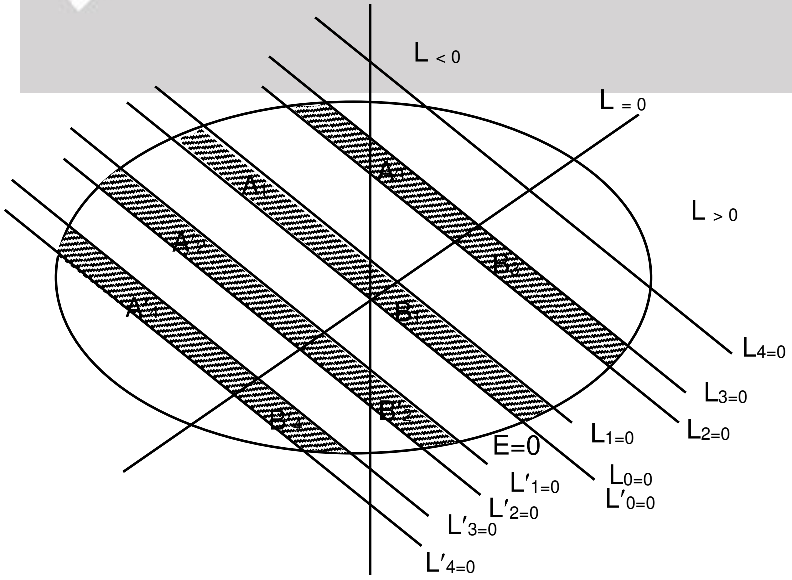
⇒ $(a_2 - a_1) > (b_2 - b_1)$

40_. Let $L = 4x - 5y$, $L_i = \frac{x}{10} + \frac{y}{8} - \frac{i}{n}$, $L'_i = \frac{x}{10} + \frac{y}{8} + \frac{i}{n}$, and $E = \frac{x^2}{50} + \frac{y^2}{32} - 1$.
Let A_i represents the area of region common between $L_{i-1} > 0$, $L_i < 0$, $E < 0$ and $L < 0$;
A'_i represents the area of region common between $L'_{i-1} < 0$, $L'_i > 0$, $E < 0$ and $L < 0$;
B_i represents the area of region common between $L_{i-1} > 0$, $L_i < 0$, $E < 0$ and $L > 0$;
B'_i represents the area of region common between $L'_{i-1} < 0$, $L'_i > 0$, $E < 0$ and $L > 0$, then value of $(A_1 + A'_2 + A_3 + A'_4 + \ldots..) + (B_1 + B'_2 + B_3 + B'_4 + \ldots..)$ is equal to.

माना कि $L = 4x - 5y$, $L_i = \frac{x}{10} + \frac{y}{8} - \frac{i}{n}$, $L'_i = \frac{x}{10} + \frac{y}{8} + \frac{i}{n}$ और $E = \frac{x^2}{50} + \frac{y^2}{32} - 1$.
माना A_i , $L_{i-1} > 0$, $L_i < 0$, $E < 0$ और $L < 0$ के मध्य उभयनिष्ठ क्षेत्र को व्यक्त करता है।
A'_i , $L'_{i-1} < 0$, $L'_i > 0$, $E < 0$ और $L < 0$ के मध्य उभयनिष्ठ क्षेत्र को व्यक्त करता है।
B_i , $L_{i-1} > 0$, $L_i < 0$, $E < 0$ और $L > 0$ के मध्य उभयनिष्ठ क्षेत्र को व्यक्त करता है।
B'_i , $L'_{i-1} < 0$, $L'_i > 0$, $E < 0$ और $L > 0$ के मध्य उभयनिष्ठ क्षेत्र को व्यक्त करता है।
तब $(A_1 + A'_2 + A_3 + A'_4 + \ldots..) + (B_1 + B'_2 + B_3 + B'_4 + \ldots..)$ का मान बराबर है।

Ans. 20 π

Sol.



$$(A_1 + A'_2 + A_3 + A'_4 + \ldots..) + (B_1 + B'_2 + B_3 + B'_4 + \ldots) = \frac{\pi ab}{2} = \frac{\pi\sqrt{50}\sqrt{32}}{2} = 20\,\pi$$